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BLOW UP OF CLASSICAL SOLUTIONS TO $\Box u = |u|^{1+\alpha}$ IN THREE SPACE DIMENSIONS

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Abstract We study the life span of classical solutions to $\Box u = |u|^{1+\alpha}$ in three space dimensions with initial data $t = 0 : u = \varepsilon f(x)$, $u_t = \varepsilon g(x)$, where f and g have compact support and are not both identically zero, ε is a small parameter. We obtain respectively upper and lower bounds of the same order of magnitude for the life span for sufficiently small ε in case $1 \le \alpha \le \sqrt{2}$. We also proved that the classical solution always blows up even when $\varepsilon = 1$ in the critical case $\alpha = \sqrt{2}$.
Key words Classical solution; life span; blow up.
Classification 35L.

1. Introduction

F. John in [1] studied the equations

$$\Box u(t,x) = |u(t,x)|^{1+\alpha}, \quad \forall (t,x) \in \boldsymbol{R}^+ \times \boldsymbol{R}^3 \tag{1.1}$$

$$t = 0 : \; u = f(x), \; u_t = g(x), \quad x \in \boldsymbol{R}^3 \tag{1.2}$$

where f and g are smooth functions with compact support and not both identically zero. He showed that the solutions must blow up if $\alpha < \alpha_0$ and global solutions exist if $\alpha > \alpha_0$ and the initial data are sufficiently small, for $\alpha_0 = \sqrt{2}$. In the case $\alpha = 1$, he also studied the life span $T(\varepsilon)$ of solutions to (1.1) with initial data

$$t = 0 : \quad u = \varepsilon f(x), \quad u_t = \varepsilon g(x) \tag{1.3}$$

By definition, the life span $T(\varepsilon)$ is sup τ, for all $\tau > 0$ such that the classical solutions to (1.1) (1.3) exist in $[0, \tau] \times \boldsymbol{R}^3$. He proved that $\varepsilon^2 T(\varepsilon)$ lies between two positive bounds when ε is small enough. For $1 < \alpha < \alpha_0$, he did not estimate the life span. In this paper we will do this:

Theorem 1.1 *Let* $T(\varepsilon)$ *be the life span of classical solutions to* (1.1)(1.3) *and* $1 \le \alpha < \alpha_0$, *then there exists a* $\varepsilon_1 > 0$ *such that for any* ε *with* $0 < \varepsilon < \varepsilon_1$

$$\kappa_2 \varepsilon^{-\alpha(\alpha+1)/(2-\alpha^2)} \le T(\varepsilon) \le \kappa_1 \varepsilon^{-\alpha(\alpha+1)/(2-\alpha^2)} \tag{1.4}$$

where κ_1 *and* κ_2 *are two positive constants independent of* ε.

After the completion of this work, we received a preprint copy of H.Lindblad's paper [4] which established a similar result as that of our theorem. His result is somewhat more precise, for he actually calculated the limit $\lim_{\varepsilon \to 0} \varepsilon^{\alpha(\alpha+1)/(2-\alpha^2)} T(\varepsilon)$. However, the critical case $\alpha = \alpha_0$ is not considered both by F.John and H.Lindblad. In this paper, by refining John's estimate we will show the following

Theorem 1.2 *The solution of* (1.1)(1.2) *must blow up in case* $\alpha = \alpha_0$.

Theorem 1.3 *Let* $T(\varepsilon)$ *be the life span of classical solution to* (1.1)(1.3) *with* $\alpha = \alpha_0$. *Then there exists a* $\varepsilon_2 > 0$ *such that for any* ε *with* $0 < \varepsilon < \varepsilon_2$

$$\exp\{\kappa_4 \varepsilon^{-\alpha_0(\alpha_0+1)}\} \le T(\varepsilon) \le \exp\{\kappa_4 \varepsilon^{-\alpha_0(\alpha_0+1)}\} \tag{1.5}$$

where κ_3, κ_4 *are two positive constants independent of* ε.

2. Preliminaries

We associate u with the solution u^0 of the linear wave equation

$$\Box u^0 = 0 \tag{2.1}$$

$$t = 0 : u^0 = f, u_t^0 = g \tag{2.2}$$

We introduce for any $(x,t) \in \boldsymbol{R}^4$, the forward and backward solid characteristic cones with vertex (x,t) restricted to the upper plane

$$\Gamma^+(x,t) = \{(y,\tau) \| y - x| \le \tau - t, \tau \ge \max(0,t)\} \tag{2.3}$$

$$\Gamma^-(x,t) = \{(y,\tau) \| y - x| \le t - \tau, \tau \ge 0\} \tag{2.4}$$

We also assume that the supports of f and g both lie in an open ball of radius ρ centered at origin.

Define

$$Lw(x,t) = \int_0^t (t-s)ds \int_{|\omega|=1} w(x + (t-s)\omega, s) ds_\omega / 4\pi \tag{2.5}$$

be the solution of

$$\Box u = w \tag{2.6}$$

with zero initial data. We associate $w(x,t)$ with the functions

$$\bar{w}(r,t) = \sup_{|x|=r} |w(x,t)|, \quad \bar{w}(r,t) = \int_{|\omega|=1} w(r\omega, t) ds_\omega / 4\pi \tag{2.7}$$

Lemma 2.1 *For* $|x| = r$, $t \ge 0$

$$|Lw(x,t)| \le \int_0^t ds \frac{1}{2r} \int_{|r-t+s|}^{r+t-s} \lambda \bar{w}(\lambda, s) ds \tag{2.8}$$

Proof See Lemma II of p. 18 in [1].

Lemma 2.2 *When* $\alpha \ge 1$, *local classical solutions of* (1.1)(1.2) *exist.*