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THE OBSTACLE PROBLEMS FOR SECOND ORDER FULLY NONLINEAR ELLIPTIC EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS[1]

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Abstract In this paper we prove the existence theorem of the strong solutions to the obstacle problems for second order fully nonlinear elliptic equations with the Neumann boundary conditions

$$F(x,u,Du,D^2u)\ge 0, x\in\Omega$$

$$u\le g, x\in\Omega$$

$$(u-g)F(x,u,Du,D^2u)=0, x\in\Omega$$

$$D_\nu u=\varphi(x,u), x\in\partial\Omega$$

where $F(x,z,p,r)$ satisfies the natural structure conditions and is concave with respect to r,p, and $\varphi(x,z)$ is nondecreasing in z, and $g(x)$ satisfies the consistency condition.
Key Words Obstacle problems; Neumann boundary conditions; logarithmic modulus of semicontinuity; global second derivative estimates.
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In paper [1] Hu Bei proved the existence theorems of the viscosity solution and the strong solution to the obstacle problems for second order nonlinear elliptic equations with the Dirichlet boundary conditions. In this paper we are concerned with the obstacle problems for second order fully nonlinear elliptic equations with the Neumann boundary conditions of the form

$$F(x,u,Du,D^2u)\ge 0,\quad x\in\Omega \tag{1}$$

$$u\le g,\quad x\in\Omega \tag{2}$$

$$(u-g)F(x,u,Du,D^2u)=0,\quad x\in\Omega \tag{3}$$

$$D_\nu u=\varphi(x,u),\quad x\in\partial\Omega \tag{4}$$

where Ω is a bounded domain in $\boldsymbol{R}^n$ with boundary $\partial\Omega\in C^4$, ν is the unit inner normal to $\partial\Omega$. $g\in C^2(\overline{\Omega})$, $\varphi\in C^2(\overline{\Omega}\times\boldsymbol{R}^1)$, $F\in C^2(\overline{\Gamma})$. Here $\Gamma=\Omega\times\boldsymbol{R}^1\times\boldsymbol{R}^n\times\$^n$, and $\$^n$ is the $n(n+1)/2$ dimensional linear space of $n\times n$ real symmetric matrices, $Du=(D_iu)$

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and $D^2u = [D_{ij}u]$ are respectively gradient vector and Hessian matrix of the function $u, D_\nu u = \nu \cdot Du$.

Letting (x, z, p, r) denote the point in Γ, we assume that F satisfies the following structure conditions:

(F1) $\lambda|\xi|^2 \le F_{r_{ij}}\xi_i\xi_j \le \Lambda|\xi|^2$, for all $\xi \in \boldsymbol{R}^n$,

(F2) $zF(x, z, 0, 0) < 0$, for all $|z| \ge z_1$,

(F3) $|F(x, z, p, 0)| \le \mu_0(|z|)(1 + |p|^2)$,

(F4) $(1 + |p|)|F_p|, |F_z|, |F_x| \le \mu_1(|z|)(1 + |p|^2 + |r|)$,

(F5) $|F_{rz}|, |F_{rx}| \le \mu_2(|z| + |p|)$,

$|F_{pz}, |F_{px}|, |F_{zz}|, |F_{zx}|, |F_{xx}| \le \mu_2(|z| + |p|)(1 + |r|)$,

(F6) F is concave with respect to r, p,

for all $(x, z, p, r) \in \Gamma$, where λ, Λ, z_1 are positive constants, μ_0, μ_1, μ_2 are nondecreasing functions.

Furthermore we assume that φ is nondecreasing in z with

(Φ1) $\varphi_z \ge 0$,

(Φ2) $\varphi(x, z_2) \le 0 \le \varphi(x, z_3)$,

for all $(x, z) \in \partial\Omega \times \boldsymbol{R}^1$ and for some two constants z_2, z_3.

At last we assume that the obstacle function g is compatible with the boundary condition, that is

(G) $D_\nu g \le \varphi(x, g)$, for all $x \in \partial\Omega$.

In order to obtain the existence theorem of the strong solutions to the obstacle problems (1), (2), (3), (4), we consider the following approximate problems by means of Lions's penalty function method

$$F(x, u, Du, D^2u) - \beta_\varepsilon(u - g) = 0, \quad x \in \Omega \tag{5}$$

$$D_\nu u = \varphi(x, u) + \varepsilon(u - g), \quad x \in \partial\Omega \tag{6}$$

where $\varepsilon \in (0, 1)$ and

$$\beta_\varepsilon(z) = \begin{cases} 0, & z \le 0 \\ z^3/\varepsilon, & z > 0 \end{cases} \tag{7}$$

From the existence theorem [2, Corollary 7.10], the problem (5), (6) has a solution $u_\varepsilon \in C^{2,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. To prove that $\{u_\varepsilon\}$ contains a converging subsequence, and the limiting function u is the strong solution of the obstacle problem (1), (2), (3), (4), we need to establish the $C^2(\overline{\Omega})$ estimates of the approximate solution u_ε, which is independent of ε. From now on we denote by C the positive constants, depending only on n, λ, Λ, μ_0, μ_1, μ_2, z_1, z_2, z_3, F, φ, g, Ω, and adopt the summation convention, i.e. the repeated indices indicate summation from 1 to n.

Theorem 1 *Let* $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *be a solution of* (5), (6), *and suppose* (F1), (F2), (Φ1), (Φ2) *hold. Then*

$$\sup_\Omega |u| \le C \tag{8}$$