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EXTERIOR PROBLEM FOR THE THREE-DIMENSIONAL EULER EQUATIONS

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Abstract *A priori* estimates for the exterior initial boundary value problems of the Euler equations are considered. The existence and uniqueness of a local solution is proved.
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0. Introduction

Let Ω be an exterior domain of $\boldsymbol{R}^3$ with bounded smooth boundary Γ. The velocity $u = (u^1(x,t), u^2(x,t), u^3(x,t))$ and the (scalar) pressure $\pi = \pi(x,t)$ of the fluid motion are assumed to be governed by the Euler equations in $Q_T = \Omega \times [0,T] (0 < T < \infty)$

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla\pi = f \\ \text{div } u = 0 \end{cases} \tag{0.1}$$

with the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega \tag{0.2}$$

and with the condition at infinity

$$\lim_{|x|\to\infty} u(x,t) = u_\infty, \quad t \in [0,T] \tag{0.3}$$

and the condition on the boundary Γ

$$u \cdot n\Big|_\Gamma = 0, \quad t \in [0,T] \tag{0.4}$$

where $f = (f^1(x,t), f^2(x,t), f^3(x,t))$ is the given external force field. $u_0(x) = (u_0^1(x), u_0^2(x), u_0^3(x))$ is the given initial velocity, $u_\infty = (u_\infty^1, u_\infty^2, u_\infty^3)$ is the given constant velocity and $u \cdot n|_\Gamma$ is the outward normal component of u on Γ.

The purpose of the present paper is to show the existence of solutions using a new priori estimate and standard technique in partial differential equations. We followed Roger Temam [1].

In the case where Ω is bounded, the existence was studied by Temam, and in the case where Ω is exterior domain in $\boldsymbol{R}^2$, Keisake Kikuchi [4] followed T.Kato [5] and gave the existence of a classical solution.

1. *A Priori* Estimates of the Solutions of the Euler Equations

1.1 Notation

We will use classical notation and results concerning the Sobolev spaces, $W^{s,p}(\Omega)$ integer, $1 \leq p < \infty$, is the Sobolev space of real valued L^p functions on Ω, such that all their derivatives to order s belong to $L^p(\Omega)$. If $p = 2$, we write $H^s(\Omega) = W^{s,2}(\Omega)$.

We write $(f, g) \cdot |f|$, the scalar product and the norm in $L^2(\Omega)$, $((f, g))_m$ and $\|f\|_m$, the scalar product and the norm in $H^m(\Omega)$,

$$((f,g))_m = \sum_{|\alpha| \leq m} (D^\alpha f, D^\alpha g)$$

where D^α is a multi-index derivation, $\alpha = \{\alpha_1, \alpha_2, \alpha_3\}$. The norm in $L^p(\Omega)$ is denoted by $|f|_p$, and $\|f\|_{m,p}$ denotes that of $W^{m,p}(\Omega)$ the same notations will be also used for the norms and scalar products in $(L^2(\Omega))^3$, $(H^m(\Omega))^3, \cdots$.

We assume that the boundary of Ω is a two-dimensional manifold of class C^r with r sufficiently large so that the usual embedding theorems hold. In particular, $W^{m,p}(\Omega) \hookrightarrow L^r(\Omega)$ where $1/r = 1/p - (m/3)$ if $m < 3/p$, $p \leq r < \infty$ is arbitrary if $m = 3/p$, $r = \infty$ if $m > 3/p$.

We recall also that if $m > 3/p$ (and Ω is smooth), $W^{m,p}(\Omega)$ is an algebra for the pointwise multiplication of functions.

Let

$$X_m = \{u \in (H^m(\Omega))^3, \ \text{div } u = 0, \quad u \cdot n = 0 \text{ on } \Gamma\}$$

$$X_{m,p} = \{u \in (W^{m,p}(\Omega))^3, \ \text{div } u = 0, \quad u \cdot n = 0 \text{ on } \Gamma\}$$

For $m = 0$, X_0 is a closed subspace of $(L^2(\Omega))^3$ and we denote by P the orthogonal projection in $(L^2(\Omega))^3$ on X_0, we recall that P is also a linear continuous operator from $(W^{m,p}(\Omega))^3$ into itself ($m \geq 1$). Indeed, if $u \in (W^{m,p}(\Omega))^3$, then $(I - P)u = \text{grad}\pi$ where π is a solution of the Neumann problem

$$\begin{aligned} \Delta\pi &= \text{div } u, \quad && \in W^{m-1,p}(\Omega) \\ \partial\pi/\partial n &= u \cdot n, \quad && \in W^{m-(1/p),p}(\Gamma) \end{aligned} \tag{0.5}$$

and $\pi \in W^{m,p}(\Omega)$ by the classed results of regularity for the Neumann problem.

We construct a function $\bar{u}(x)$ satisfying

$$\text{div } \bar{u}(x) = 0$$

$$\bar{u}(x) = 0, \quad x \in \Gamma$$

$$\lim_{|x| \to \infty} \bar{u}(x) = u_\infty$$

For example, we select $\bar{u}(x) = u_\infty - b(x)$, $b(x) = \text{rot}(\xi(x)d(x))$, $d(x) = (u_\infty^2 x_3, u_\infty^3 x_1, u_\infty^1 x_2)$, $\xi(x)$ is a smooth "cut-off" function, which is equal to 1 nearby Γ, and equal to 0 when $|x|$ is very large.