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A GENERALIZATION OF EELLS-SAMPSON'S THEOREM

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Abstract We generalize the well-known Eells-Sampson's theorem on the global existence and convergence for the heat flow of harmonic maps. The assumption that the curvature of the target manifold N be nonpositive is replaced by the weaker one requiring that the universal cover $\tilde{N}$ admit a strictly convex function with quadratic growth.

Key Words Harmonic maps; heat flow.
Classifications 58E20, 58G11.

1. Introduction

Let (M,g) and (N,h) be two compact Riemannian manifolds without boundary. It will be convenient to embed (N,h) isometrically into some Euclidean space $\boldsymbol{R}^K$ so that we may consider N as a submanifold of $\boldsymbol{R}^K$ with the induced metric. Let

$$C^1(M,N)=\{u=(u^1,\cdots,u^K)\in C^1(M,\boldsymbol{R}^K): u(x)\in N\ \forall x\in M\}$$

For any $u\in C^1(M,N)$, the energy density of u is defined by

$$e(u)=\frac{1}{2}|\nabla u|^2=\frac{1}{2}\sum_{\alpha,\beta=1}^{m}\sum_{i=1}^{K}g^{\alpha,\beta}\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^i}{\partial x^\beta}$$

where $m=\dim M$. The energy of u is given by

$$E(u)=\int_M e(u)dv_g$$

By definition, harmonic maps are critical points of the energy $E(u)$ as a functional on $C^1(M,N)$. If u is harmonic then it satisfies the Euler-Lagrange equation

$$\tau(u)^i\equiv\Delta u^i-\sum_{\alpha,\beta}\sum_{j,k}g^{\alpha,\beta}A^i_{j,k}(u)\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta}=0$$

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where $A(y)$ is the second fundamental form of N in R^K at $y \in N$. We call $\tau(u)$ the tension field of u.

J. Eells and J. H. Sampson [5] first introduced the method of heat flow in the study of the existence problem for harmonic maps. This flow is a negative gradient flow for the energy $E(u)$, defined by the parabolic equation

$$\frac{\partial u}{\partial t} = \tau(u), \quad u(0,x) = u_0(x) \tag{1.1}$$

It is shown in [5] that for $u_0 \in C^1(M,N)$ (1.1) has a unique solution $u \in C^\infty((0,T) \times M, N) \cap C^0([0,T) \times M, N)$, where $T = T(u_0) \in (0,\infty]$ is the maximal existence time for the solution u. If $T < \infty$, the solution must blow up in the sense that

$$\lim_{t \to T} \|\nabla u\|_{C^0(M)} = \infty$$

If $T = \infty$, we say that the solution exists globally. Eells and Sampson proved that if the target manifold N has non-positive curvature then the solution is global and it converges (in $C^k(M,N)$ for any $k \geq 1$) to a harmonic map as $t \to \infty$.

It is our aim in this note to present a generalization of the above stated theorem of Eells-Sampson. Our result is as follows.

Theorem 1.1 *Let $(\tilde{N}, \tilde{h})$ be the universal covering of (N,h). Suppose that $\tilde{N}$ admits a strictly convex function $\rho \in C^2(\tilde{N})$ with quadratic growth, i.e. ρ satisfies*

$$\nabla^2 \rho \geq c_0 \tilde{h} \text{ on } \tilde{N} \tag{1.2}$$

where $c_0 > 0$ is a constant, and

$$0 \leq \rho(y) \leq c_1 d^2_{\tilde{N}}(y, y_0) + c_2, \quad \forall y \in \tilde{N} \tag{1.3}$$

where $c_1, c_2 > 0$ are constants, $y_0 \in \tilde{N}$ and $d_{\tilde{N}}$ is the distance on $\tilde{N}$. Then every solution u of (1.1) is global and subconverges to some harmonic map as $t \to \infty$. Moreover, we have the following estimate

$$\|u(t,\cdot)\|_{C^1(M,N)} \leq C(E(u_0)), \quad t \geq 1 \tag{1.4}$$

Remark 1.1 If the sectional curvature of N is nonpositive then it is known that $\rho \equiv d^2_{\tilde{N}}(\cdot, y_0)$ is strictly convex, hence (1.2) and (1.3) are satisfied. Consequently, Eells-Sampson's theorem follows from our Theorem 1.1. The quadratic growth condition (1.3) seems to be a technical one. We doubt its necessity. It would be interesting if one can remove this condition from the theorem. For the case where the initial map is homotopic to a constant map, the condition (1.3) can actually be removed (cf. Remark 3.1). But, in such a case the solution u of (1.1) has to converge to a constant map.

Remark 1.2 The assumption of the existence of a strictly convex function on $\tilde{N}$ is already sufficient for the existence of a smooth minimizing harmonic map in each