INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 10, Number 2, Pages 411–429

RESIDUAL-BASED A POSTERIORI ESTIMATORS FOR THE T/Ω MAGNETODYNAMIC HARMONIC FORMULATION OF THE MAXWELL SYSTEM

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(Communicated by Mark Ainsworth)

Abstract. In this paper, we focus on an a posteriori residual-based error estimator for the $\mathbf{T}/\Omega$ magnetodynamic harmonic formulation of the Maxwell system. Similarly to the $\mathbf{A}/\varphi$ formulation [7], the weak continuous and discrete formulations are established, and the well-posedness of both of them is addressed. Some useful analytical tools are derived. Among them, an ad-hoc Helmholtz decomposition for the $\mathbf{T}/\Omega$ case is derived, which allows to pertinently split the error. Consequently, an a posteriori error estimator is obtained, which is proven to be reliable and locally efficient. Finally, numerical tests confirm the theoretical results.

Key words. Maxwell equations, potential formulations, a posteriori estimators, finite element method.

1. Introduction

Let us consider the electromagnetic fields, modeled by the well-known Maxwell system :

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1}$$

$$\operatorname{curl} \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \tag{2}$$

where $\mathbf{E}$ is the electrical field, $\mathbf{H}$ the magnetic field, $\mathbf{B}$ the magnetic flux density, $\mathbf{J}$ the current flux density (or eddy current) and $\mathbf{D}$ the displacement flux density. Equation (1) is classically called Maxwell-Faraday equation and equation (2) Maxwell-Ampère one. In the low frequency regime, the quasistatic approximation can be applied, which consists in neglecting the temporal variation of the displacement flux density with respect to the current density [12], such that the propagation phenomena are not taken into account. Consequently, equation (2) becomes :

$$\operatorname{curl} \mathbf{H} = \mathbf{J}. \tag{3}$$

Here, the current density $\mathbf{J}$ can be decomposed in two terms such that $\mathbf{J} = \mathbf{J}_s + \mathbf{J}_{ec}$. $\mathbf{J}_s$ is a known distribution current density generally generated by a coil, and $\mathbf{J}_{ec}$ represents the eddy current. Both equations (1) and (3) are linked by the material constitutive laws :

$$\mathbf{B} = \mu \mathbf{H}, \tag{4}$$

$$\mathbf{J}_{ec} = \sigma \mathbf{E}, \tag{5}$$

where μ stands for the magnetic permeability and σ for the electrical conductivity of the material. Figure 1 displays the domains configuration we are interested

Received by the editors and, in revised form, March 5, 2012.
2000 *Mathematics Subject Classification.* 35Q61, 65N30, 65N15, 65N50.

in. We consider an open connected domain $D \subset \mathbb{R}^3$, with a Lipschitz boundary $\Gamma = \partial D$. We define an open simply connected conductor domain $D_c \subset D$ and we note $\Gamma_c = \partial D_c$ its boundary which is supposed to be Lipschitz and such that $\Gamma_c \cap \Gamma = \emptyset$. In D_c, the electrical conductivity σ is not equal to zero so that eddy currents can be created. Finally we define $D_e = D \backslash \overline{D_c}$ as the part of D where the electrical conductivity σ is equal to zero. Boundary conditions associated with the system (1)-(3) are given by:

$$\mathbf{B}.\mathbf{n} = 0 \text{ on } \Gamma, \tag{6}$$

$$\mathbf{J}_{ec}.\mathbf{n} = 0 \text{ on } \Gamma_c. \tag{7}$$

We intend to solve this problem by using the potential formulations often used for

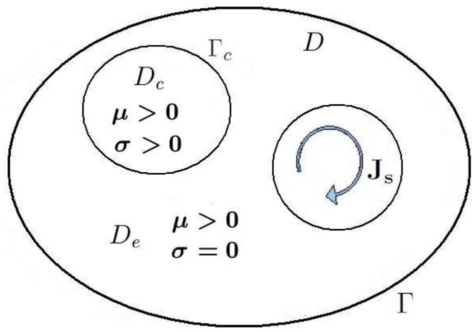

FIGURE 1. Domains configuration.

electromagnetic problems. A similar work was already done for the so-called $\mathbf{A}/\varphi$ formulation [7]. Another recent paper was concerned with the $\mathbf{A}/\varphi$ formulation [3] but in a different framework, having at last the potential vector $\mathbf{A}$ as an unique unknown, and considering the case where μ is constant. Here, we consider the $\mathbf{T}/\Omega$ formulation which is first described. Since $\operatorname{div} \mathbf{J}_s = 0$ in D, there exists a source magnetic field $\mathbf{H}_s$ such that [9]:

$$\operatorname{curl} \mathbf{H}_s = \mathbf{J}_s \text{ in } D,$$

and since the conductor domain D_c is simply connected, as $\operatorname{div} \mathbf{J}_{ec} = 0$, there exists a source magnetic field $\mathbf{T}$ such that:

$$\operatorname{curl} \mathbf{T} = \mathbf{J}_{ec} \text{ in } D_c.$$

From (3), a magnetic scalar potential Ω can be introduced so that the magnetic field $\mathbf{H}$ can be written by:

$$\mathbf{H} = \begin{cases} \mathbf{H}_s + \mathbf{T} - \nabla\Omega & \text{in} \quad D_c, \\ \mathbf{H}_s - \nabla\Omega & \text{in} \quad D_e. \end{cases} \tag{8}$$

From (4), (5) and (8), equation (1) becomes:

$$\operatorname{curl}\left(\frac{1}{\sigma}\operatorname{curl}\mathbf{T}\right) + \frac{\partial}{\partial t}\left(\mu\left(\mathbf{T} - \nabla\Omega\right)\right) = -\frac{\partial}{\partial t}\left(\mu\mathbf{H}_s\right) \text{ in } D_c. \tag{9}$$

Consequently, we also have

$$\operatorname{div}\left(\mu\left(\mathbf{T} - \nabla\Omega\right)\right) = -\operatorname{div}\left(\mu\mathbf{H}_s\right) \text{ in } D_c. \tag{10}$$