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BACKLUND TRANSFORMATIONS FOR THE ISOSPECTRAL AND NON-ISOSPECTRAL MATRIX KDV HIERARCHIES

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Abstract By transforming the usual Lax pairs of isospectral and non-isospectral matrix Kdv hierarchies into Lax pairs Riccati form, a unified explicit from of Backlund transformations and superposition formulas for these two kinds of hierarchies of equations can be obtained.
Key Words Backlund transformation; noncommutativity of matrix; Lax pairs.
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1. Introduction

It is well-known that Backlund transformation (BT) enables us to obtain nontrivial solutions of nonlinear evolution equation (NLEE) from their trivial solution, so it is very important to search for BT for NLEEs. Since the seventies, people have proposed various methods to obtain BTs for NLEEs (such as: Inverse method, Darboux transformation, Hirota method and so on.)· Among them the Darboux transformation method is the most effective one, which makes it possible to obtain explicit BTs for some whole hierarchies of isospectral NLEEs.

However, till now people still have little known about BTs for hierarchies of non-isospectral and matrix NLEEs, since it is very difficult to apply the above methods directly to these NLEEs. Recently, in [2] the authors successfully obtained an explicit form of BTs for the isospectral and nonisospectral scalar Kdv hierarchies, the approach is to convert the usual Lax pairs of these NLEEs to Lax pair of Riccati form, and then uses some invariabilities of these new Lax pairs. The process is very simple and clear.

In this paper we employ this method to study the isospectral and non-isospectral matrix Kdv hierarchies. The main difficulty lies in the non-commutativity of matrices, which makes the process of obtaining BTs much more complicated and difficult. As for the scalar case, we also obtain an explicit form of BTs for the isospectral and non-isospectral matrix Kdv hierarchies, due to our introduction of some matrix operators.

2. Backlund Transformations for the Isospectral Matrix Kdv Hierarchies

Consider the isospectral matrix Kdv hierarchy.

$$U_t = \mathcal{L}^n U_x, \qquad n = 0, 1, 2, \cdots \tag{2.1}$$

where U is a $N \times N$ matrix function defined on $R^2(x,t)$. In this paper we always assume that U and its any order x derivatives tend to zero rapidly when $x \to -\infty$, $\mathcal{L}$ is defined by

$$\mathcal{L} = -D^2 + 2\overline{U} + \overline{U}_x D^{-1} - \overline{\overline{U}} D^{-1} \overline{\overline{U}} D^{-1} \tag{2.2}$$

where operators $\overline{U}$ and $\overline{\overline{U}}$ are defined by

$$\overline{U} \cdot S = US + SU = \{U \cdot S\}, \qquad \overline{\overline{U}} \cdot S = US - SU = [U \cdot S]$$

for any $N \times N$ matrix function S, and $D^{-1} = \int_{-\infty}^{x} d_x, D = \frac{d}{d_x} I$. We can show that Equation (2.1) has the following Lax pair

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_x = \begin{pmatrix} 0 & I \\ U - I & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{2.3a}$$

$$\begin{pmatrix} y \\ y \end{pmatrix}_t = \begin{pmatrix} A & B \\ C & E \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{2.3b}$$

where

$$B = \sum_{j=0}^{n} B_j (4\lambda)^{n-j}, \qquad B_0 = I$$

$$B_1 = 2U, \qquad B_{j+1} = \mathcal{L}^j DB, \qquad j = 1, 2 \cdots n-1$$

$$A = \frac{1}{2} D^{-1} [U \cdot B] - \frac{1}{2} B_x, \qquad C = \frac{1}{2} \{U \cdot B\} - \frac{1}{2} B_{xx} - \lambda B$$

$$E = \frac{1}{2} D^{-1} [U \cdot B] + \frac{1}{2} B_x, \qquad \lambda_t = 0$$

and

Let $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be a fundamental solution matrix of Equation (2.3). Define $y = y_2 y_1^{-1}$ then (2.3) becomes

$$y_x = U - \lambda I - y^2, \qquad y_t = C + Ey - yA - yBy \tag{2.4a}$$

From the boundary conditions of U, we can assume that the y tends to $\pm\sqrt{-\lambda I}$ when $x \to -\infty$ and its any order x derivatives tend to zero when $x \to -\infty$. We call Equations (2.4a) and (2.4b) the Lax pair of Riccati form for Equation (2.1) to obtain BT for Equation (2.1), we need some invariabilities of Equation (2.4a). So we must put this equation in a simpler and clearer form. For this purpose we need some lemmas.

Lemma 3.1 *Define:*

$$\Psi(y) = D + \bar{y}$$

$$\Phi(y, \lambda) = -D^2 + 2\bar{y}^2 + \bar{y}_x D^{-1} \bar{y} + 4\lambda I - \bar{\bar{y}} D^{-1} \bar{\bar{y}}^2 D^{-1} \bar{y} - \bar{\bar{y}} D^{-1} \bar{\bar{y}}_x$$

then

$$D \cdot U = \Psi y_x, \qquad U_t = \Psi y_t + \lambda_t I$$
$$\Psi(y) \Phi(y, \lambda) = \mathcal{L}(U) \Psi(y) \tag{2.5}$$

Proof direct calculation.

Lemma 2.2 *Define*

$$P = -D^2 + D\bar{y} + \bar{y}^2 - \bar{\bar{y}} D^{-1} \bar{\bar{y}}_x - 2y\bar{\bar{y}} - \bar{\bar{y}} D^{-1} \bar{\bar{y}}^2 \tag{2.6}$$