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EXISTENCE OF VISCOSITY SOLUTIONS OF SECOND ORDER FULLY NONLINEAR ELLIPTIC EQUATIONS[①]

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Abstract We consider the problem of existence for viscosity solutions of second order fully nonlinear elliptic partial differential equations $F(D^2u, Du, u, x) = 0$. We prove existence results for viscosity solutions in $W^{1,\infty}$ under assumptions that function F satisfies the natural structure conditions. We do not assume F is convex.

Key Words Viscosity solutions; Second order fully nonlinear elliptic equations; Existence.

Classification 35J60.

1. Introduction

This paper deals with the problem of existence for solutions of second order fully nonlinear elliptic equations

$$F(D^2u, Du, u, x) = 0 \quad \text{in } \Omega \tag{1.1}$$

with Dirichlet boundary condition

$$u = g \quad \text{on } \partial\Omega \tag{1.2}$$

where Ω is a bounded domain in R^n with $C^{1,1}$ boundary. Here F is a real function on $\Gamma = S(n) * R^n * R * \Omega$, $S(n)$ denotes the $n * n$ real valued symmetric matrices, and Γ' will denote set $S(n) * R^n * R$. We assume g is a C^2 real function on $\overline{\Omega}$.

The existence results for such problems depend on both the properties of the function F and the space in which solutions are taken. Using the method of continuity, we can establish existence result for classical solutions of (1.1) and (1.2) under some conditions on F which include the convexity of F. Otherwise, some existence results of $W^{2,p}$ solutions of (1.1) and (1.2) can be obtained: for F "linear at infinity" ([6]), for F "close to linear" ([8]).

The definition of "viscosity solution" was introduced by [4] as a notion of weak solution for H-J equations in 1983. Under some assumptions, the uniqueness and exis-

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tence of viscosity solutions can be established. In [10] the definition of viscosity solution was extended to second order problems, and if F is convex, the uniqueness of viscosity solutions was proved. In 1986, R. Jensen [9] proved uniqueness of viscosity solutions for (1.1) and (1.2). He does not assume F is convex and not allow spatial dependence in x. We extended the result of [9] to the case that F can be dependent on x but we must assume F is uniformly continuity in x ([2]).

In this paper, we prove the following existence theorem.

Theorem *Let $F\in C^3(\Gamma)$ satisfy natural structure conditions and the following condition*

$$|F_{rx}|, |F_{rxx}|, |F_{px}|, |F_{pxx}|, |F_x|, |F_{xx}|, |F_{xxx}| \leqslant C(1+|p|^2+|r|)$$

and suppose that $g\in C^2(\overline{\Omega})$. Then there exists a $W^{1,\infty}(\Omega)$ viscosity solution for problem (1.1) *and* (1.2).

The method we use in the proof of the above theorem involves solving a sequence of approximate problems by the m-accretive operator technique, making $W^{1,\infty}$ estimates for $W^{2,p}$ ($p>2n$) solutions and passing to limits by means of a modification of G. Minty's Hilbert space method.

2. Preliminaries

We begin by some definitions.

Definition 2.1 *Let $u\in C(\overline{\Omega})$, the superdifferential $D^+u(x)$ (subdifferential $D^-u(x)$) is defined as the set*

$$D^+u(x)=\{(p,M)\in R^n * S(n): u(x+z) \leqslant u(x)+p*z+((M/2)*z,z)+o(|z|^2)\}$$

$$(D^-u(x)=\{(p,M)\in R^n * S(n): u(x+z) \geqslant u(x)+p*z+((M/2)*z,z)+o(|z|^2)\})$$

Definition 2.2 *$u\in C(\Omega)$ is a viscosity supersolution (subsolution) of* (1.1) *if*

$$F(M,p,u(x),x)\leqslant 0 \quad \text{for all } (p,M)\in D^-u(x), x\in\Omega$$

$$(F(M,p,u(x),x)\geqslant 0 \quad \text{for all } (p,M)\in D^+u(x), x\in\Omega)$$

$u\in C(\overline{\Omega})$ is a viscosity solution of (1.1) *if it is both a viscosity supersolution and subsolution.*

For superdifferential and subdifferential, we have (see [6])

Lemma 2.3 *Suppose $u\in W^{1,p}_g(\Omega)$ $(p>n)$ and let $x_0\in\Omega$. Then for any pair $(p,M)\in D^-u(x_0)$ (or $D^+u(x_0)$), there exists a sequence $\{\varphi_k\}\subset C^\infty_g(\Omega)$ such that*

(i) $\varphi_k(x_0)\to u(x_0)$, $D\varphi_k(x_0)\to p$, $D^2\varphi_k(x_0)\to M$

(ii) $\varphi_k(x_0)-u(x_0)=\|\varphi_k-u\|_{C(\overline{\Omega})}>\varphi_k(x)-u(x)$

(*or* $u(x_0)-\varphi_k(x_0)=\|u-\varphi_k\|_{C(\overline{\Omega})}>u(x)-\varphi_k(x)$)