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DEGENERATE NONLINEAR BOUNDARY VALUE PROBLEMS FOR FULLY NONLINEAR ELLIPTIC EQUATIONS OF SECOND ORDER*

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Abstract In this paper, we will consider the degenerate nonlinear boundary value problems for nonlinear elliptic equations. We extend Lieberman & Trudinger's results from nondegenerate oblique derivative boundary values to degenerate boundary values.

Key Words Degenerate; nonlinear; estimate; elliptic equation.
Classifications 35J25; 35J65.

0. Introduction

In this paper we are concerned with degenerate nonlinear boundary value problems for fully nonlinear elliptic equations of the general form,

$$F[u] = F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega \tag{0.1}$$

$$G[u] = G(x, u, Du) = 0 \quad \text{on } \partial\Omega \tag{0.2}$$

where $\Omega \subset R^n$ is a bounded smooth domain, $Du = (D_i u)$, $D^2u = (D_{ij}u)$ denote the gradient and Hessian matrix of the real-valued function u, $F(x, z, p, r)$ and $G(x, z, p)$ are C^2 and C^3 real-valued functions on $\Gamma = \Omega \times R \times R^n \times \$^n$ and $\Gamma' = \partial\Omega \times R \times R^n$ respectively. Here $\$^n$ denotes the $n(n+1)/2$ dimensional linear space of $n \times n$ real symmetric matrices.

The statement that the operator F is elliptic on Γ means that for all $(x, z, p, r) \in \Gamma$, the matrix

$$F_r(x, z, p, r) = [F^{ij}(x, z, p, r)] = \left[\frac{\partial F}{\partial r_{ij}}(x, z, p, r)\right]$$

is positive.

We introduce certain natural structure conditions to which F and G are subject. Letting $\lambda(x, z, p, r)$ and $\Lambda(x, z, p, r)$ denote the minimum and maximum eigenvalues of

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$F_r(x,z,p,r)$, letting μ_i, $\phi_i (i = 0, 1, \cdots, 4)$ denote non-decreasing real positive functions and $\alpha_0, \alpha_1, \beta_0$ H_0 be positive constants, we formulate the conditions as follows:

For all $(x,z,p,r) \in \Gamma$,

(F$_1$) $1/\mu_0(|z|+|p|) \le \Lambda(x,z,p,r) \le (1+|z|^{\theta_0})\mu_0(|p|), (0<\theta_0<1)$,

(F$_2$) $\Lambda(x,z,p,r) \le \mu_1(|z|)\lambda(x,z,p,r)$, (uniform ellipticity),

(F$_3$) $|F(x,z,p,0)| \le \lambda\mu_2(|z|)(1+|p|^2)$,

(F$_4$) $|F_z|, |F_x| \le \lambda\mu_3(|z|)(1+|p|^2+|r|)$,

$$(1+|p|)|F_p| \le \alpha_1\Lambda(1+|p|^2+|r|),$$

$$F_z(x,z,0,0) \le -\alpha_0,$$

(F$_5$) $(1+|r|)(|F_{rx}|+|F_{rz}|+|F_{rp}|), |F_{pp}|, |F_{pz}|, |F_{px}|, |F_{zz}|, |F_{zx}|, |F_{xx}|$
$\le \lambda\mu_4(|z|+|p|)(1+|r|)$,

(F$_6$) $F_{rr} \le 0$.

For all $(x,z,p) \in \Gamma'$

(G$_1$) $0 \le (G_p(x,z,p),\gamma(x)) \le |G_p(x,z,p)| \le \phi_0(|z|)(G_p(x,z,p),\gamma(x))$,

(G$_2$) $G_z \le 0, -(G_p(x,z,p),\gamma(x)) + G_z(x,z,p) \le -\beta_0$,

(G$_3$) $|G_p|, |G_{px}|, |G_{pxz}|, |G_{pz}|, (1+|p|)|G_{pp}| \le \phi_1(|z|)$,

(G$_4$) $|G|, |G_z|, |G_{zz}|, |G_{zzz}|, |G_x|, |G_{xz}|, |G_{xx}| \le \phi_2(|z|)(1+|p|)$,

(G$_5$) $|G_p(x,z,p)| \le \phi_3(|z|+|\tilde{z}|)|G_p(x,\tilde{z},\tilde{p})|, \forall (x,\tilde{z},\tilde{p}) \in \Gamma'$,

(G$_6$) $|G_x(x,z,p)| \le \phi_4(|z|)\Big[1+(G_p,\gamma)|p| + \sqrt{(G_p,\gamma)}\,|p|$

$$+\Big(\sum_{i=1}^{n} \hat{G}_{p_i}(x,z,p)p_i + \delta(p,\gamma)\Big)/|\hat{G}_p + \delta\gamma|\Big] + (1-\theta_1)(-G_z)|p'|,$$

$\forall \delta \ne 0, (0 < \theta_1 \le 1)$.

(G$_7$) There exist functions $Z^{\pm}(x)$ with $|Z^{\pm}(x)| \le H_0/2$, $x \in \partial\Omega$, such that

$$G(x, Z^-(x), (H_0 - Z^-(x))\gamma(x)) \ge 0,$$
$$G(x, Z^+(x), (Z^+(x) - H_0)\gamma(x)) \le 0, \quad \forall\, x \in \partial\Omega,$$