J. Partial Differential Equations
Vol. 3, No. 1(1990), 1—12

PERIODIC SOLUTIONS OF NONLINEAR WAVE EQUATIONS WITH DISSIPATIVE BOUNDARY CONDITIONS

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(Received Sept. 30, 1986; revised June 20, 1989)

Abstract Applying Nash-Moser's implicit function theorem, the author proves the existence of periodic solution to nonlinear wave equation

$$u_{tt} - u_{xx} + \varepsilon g(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0$$

with a dissipative boundary condition, provided ε is sufficiently small.

Key Words Nonlinear wave equation; time periodic solution; dissipative boundary condition.

Classifications 35L70; 35L20; 35B10.

0. Introduction

In this paper we discuss the existence of time-periodic solution for the following boundary value problem of nonlinear wave equation

$$F_\varepsilon(u) \equiv u_{tt} - u_{xx} + \varepsilon g(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0 \tag{0.1}$$

$$u(t, 0) = 0 \tag{0.2}$$

$$u_x(t, l) + \lambda u_t(t, l) = 0 \tag{0.3}$$

where g is periodic in t with period ω, $\lambda \neq 0$ is a constant. When (2) and (3) are Dirichlet's boundary conditions and there is a dissipative term λu_t in operator $F_\varepsilon(u)$, Rabinowitz in [1] proved the existence of periodic solution if ε is sufficiently small. The aim of this paper is to prove that if a dissipative boundary condition is given at one end of the interval instead of the term λu_t in the equation, then the problem admits a periodic solution provided ε is sufficiently small.

As the second order derivatives of u appear in the nonlinear term of the operator $F_\varepsilon(u)$, we shall use Nash-Moser's implicit function theorem to obtain the periodic solution of the problem. We shall apply the version of this theorem given by Moser in [2] which requires to solve the linearized equation only.

1. The Main Results

All functions mentioned in this section are periodic in t with the period ω. For simplicity, ω, l and λ are taken below to be 2π, 1 and 1, respectively.

Set

$$Q = [0, 2\pi] \times [0, 1]$$

and

$$U_p = \{u \mid \partial_t^j u \in H^5(Q), j \leqslant p, u(t, 0) = u_x(t, 1) + u_t(t, 1) = 0\},$$

$$F_p = \{u \mid \partial_t^j u \in H^3(Q), j \leqslant p\}$$

with norms

$$\| u \|_{U_p} = \max_{0 \leqslant j \leqslant p} \| \partial_t^j u \|_{H^5(Q)}$$

$$\| u \|_{F_p} = \max_{0 \leqslant j \leqslant p} \| \partial_t^j u \|_{H^3(Q)}$$

where $H^s(Q)$ are Sobolev's spaces on Q with norms $\| \cdot \|_{H^s(Q)}$. It is clear that U_p, F_p are Banach's spaces and

$$U_0 \supset U_1 \supset U_2 \supset \cdots, \quad F_0 \supset F_1 \supset F_2 \supset \cdots$$

For any $u \in U_p$, we can write it in the form of a Fourier's series

$$u = \frac{1}{2\pi} a_0(x) + \frac{1}{\pi} \sum_{j=1}^{\infty} (a_j(x) \cos jx + b_j(x) \sin jx)$$

We define the truncation operator T_N by

$$T_N u = \frac{1}{2\pi} a_0(x) + \frac{1}{\pi} \sum_{j \leqslant N} (a_j(x) \cos jx + b_j(x) \sin jx)$$

then it is easy to prove that

$$T_N: \quad U_r \longrightarrow U_{r+s}$$

and

$$\| T_N u \|_{U_{r+s}} \leqslant N^s \| u \|_{U_r} \tag{1.1}$$

$$\| (I - T_N) u \|_{U_r} \leqslant N^{-s} \| u \|_{U_{r+s}} \tag{1.2}$$

for any nonnegative integer N.

The linearized operator of the nonlinear operator $F_\varepsilon(u)$ is

$$F_\varepsilon'(u) v \equiv v_{tt} - v_{xx} + \varepsilon(a_{11} v_{tt} + a_{12} v_{tx} + a_{22} v_{xx} + a_1 v_t + a_2 v_x + a_0 v)$$

where $a_0 = g_u(t, x, u_{tt}, \cdots, u)$, $a_1 = g_{u_t}$, $a_2 = g_{u_x}$, $a_{11} = g_{u_{tt}}$, $a_{12} = g_{u_{tx}}$, $a_{22} = g_{u_{xx}}$.

In the following sections, we shall prove that if ε is sufficiently small, then $F_\varepsilon(u)$ satisfies the following for a constant b:

(1) If $u \in U_2$ and $\| u \|_{U_2} \leqslant b^{-1}$, then $F_\varepsilon(u) \in F_2$ and $F_\varepsilon'(u)$ is linear and bounded from U_2 into F_2.

(2) For any $u, v \in U_2$ and $\| u \|_{U_2} \leqslant b^{-1}$, $\| u + v \|_{U_2} \leqslant b^{-1}$, we have $\| F_\varepsilon(u+v) - F_\varepsilon(u) - F_\varepsilon'(u) v \|_{F_2} \leqslant b \| v \|_{U_2}^2$

(3) If $u \in U_{2+k}$ and $N \geqslant 1$ satisfy

$$\| u \|_{U_{2+\lambda}} \leqslant b^{-1} N^\lambda \quad \text{for} \quad \lambda = 0, 1, \cdots, k$$

then

$$\| F(u) \|_{F_{2+\lambda}} \leqslant b N^\lambda \quad \text{for} \quad \lambda = 0, 1, \cdots, k$$

(4) If $u \in U_{2+k}, h \in F_{2+k}$ and $N \geqslant 1$ satisfy

$$\| u \|_{U_{2+\lambda}} \leqslant b^{-1} N^\lambda, \quad \| h \|_{F_{2+\lambda}} \leqslant b N^\lambda \quad \text{for} \quad \lambda = 0, 1, \cdots, k$$

then the linearized equation

$$F_\varepsilon'(u) v = h \tag{1.3}$$

admits a solution $v \in U_k$ satisfying

$$\| v \|_{U_0} \leqslant b \| h \|_{F_2},$$