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THE CONDITIONS FOR SOME LINEAR PARTIAL DIFFERENTIAL EQUATIONS TO BE SOLVABLE IN $\mathscr{S}'$

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Abstract Using Bargmann's transformation and some basic results of theory of analytic functions with several complex variables, we have disscussed two classes of LPDOs in this paper. We prove that each operator of one class of them is surjective both from $\mathscr{S}'$ to $\mathscr{S}'$ and from L^2 to L^2, but not injective, and each operator of another class is injective from $\mathscr{S}'$ to $\mathscr{S}'$ but not surjective. And in the latter case, the necessary and suffcient conditions for the corresponding equations to be solvable in $\mathscr{S}'$ are given.

Key Words Solvability; Hermite expansion; Bargmann Space.
Classifications 35A; 35D; 35G.

0. Introduction

Let $P(x, D_x)$ be a linear partial differential operator with polynomial coefficients, and let $\mathscr{S}'$ be the space of temperate distributions on R^n, we consider the equation

$$P(x, D_x)u = f \tag{0.1}$$

If (0.1) has a solution $u \in \mathscr{S}'$ for given $f \in \mathscr{S}'$, we call (0.1) solvable in $\mathscr{S}'$. $P(x, D_x)$ will be called a solvable operator if $P(x, D_x)$ is a mapping from $\mathscr{S}'$ onto itself.

The problem on solvability in $\mathscr{S}'$ was paid great attention to long ago. Hörmander and Lojasiewicz first proved the existence of fundamental solutions in $\mathscr{S}'$ for equations with constant coefficients respectively in [1] and [2]. Since then, several mathematicians have simplified or improved the proofs (see [3], [4] and [5]).

It may be worth to point out that the problems on the local solvability and the hypoellipticity of left (right) invariant differential operators on nilpotent Lie groups, by means of their unitary representations, can be reduced to ones on solvability and uniqueness of solutions of (0.1) in $\mathscr{S}'$, as be shown in some recent works (see Section2 and Section 4 of Chapter 2 in [6]). Therefore it seems to be reasonable that study of solvability and uniqueness in $\mathscr{S}'$ will be paid much attention to. Just because of this background, we made a systematic study on the problem for a class of LPDOs in [7]. In this paper, other two classes of LPDOs are discussed. We find that each operator of the first class is surjective both from $\mathscr{S}'$ to $\mathscr{S}'$ and from L^2 to L^2, but not injective. And each operator of another class is injective from $\mathscr{S}'$ into $\mathscr{S}'$, but not surjective. In the latter case, the necessary and sufficient conditions for the corresponding equations to be solvable in $\mathscr{S}'$ are given.

The main tools used in this paper are Bargmann's transformation and some basic results of theory of analytic functions with several complex variables.

1. A Class of Unsolvable Operators

Let

$$E_j = \frac{1}{\sqrt{2}}(x_j - \partial_j), \quad j = 1,2,\cdots,n$$

and

$$E^\alpha = \prod_{j=1}^{n} E_j^{\alpha_j}, \quad \alpha = (\alpha_1,\cdots,\alpha_n) \in I_+^n$$

where I_+ is the set of nonnegative integers and $I_+^n = \overbrace{I_+ \times \cdots \times I_+}^{n}$.

Let

$$B(x,\partial_x) = b(E) = \sum_{|\alpha|=m} b_\alpha E^\alpha \tag{1.1}$$

with complex constants b_α.

It is clear that $b(E)$ is a linear partial differential operator whose coefficients are polynomials of x with the principal part of constant coefficient. We shall prove that $b(E)$ is not solvable in $\mathscr{S}'$ and give necessary and sufficient conditions for the equation

$$b(E)u = f, \quad f \in \mathscr{S}' \tag{1.2}$$

to have a solution in $\mathscr{S}'$.

Since the Fourier transformation makes $B(E)$ be turned into $\sum_{|\alpha|\leqslant m} (i)^{|\alpha|} b_\alpha E^\alpha$ which is still of the same kind as $b(E)$, it is no use for solving our problem. Therefore we shall introduce the following Bargmann transformation instead of the Fourier transformation.

Let $\mathfrak{Z}$ be the space of holomorphic functions on C^n. For given real number k, set

$$\mathfrak{Z}^k = \left\{ f : f \in \mathfrak{Z}, |f|_k = \left(\int |f(z)|^2 (1 + |z|^2)^k du(z) \right)^{1/2} < +\infty \right\}$$

where $du(z) = \pi^{-n} e^{-|z|^2} d^n z$ with $d^n z = \prod_{j=1}^{n} d\xi_j d\eta_j$, where $z = \xi + i\eta$. Put $\mathfrak{Z}^{+\infty} = \bigcup_k \mathfrak{Z}^k$ and $\mathfrak{Z}^{-\infty} = \bigcap_k \mathfrak{Z}^k$.

Let

$$A(z,x) = \pi^{-n/4} \exp\left[-\frac{1}{2}(z^2 + x^2) + \sqrt{2}\, z \cdot x \right], \quad \forall z \in C^n, x \in R^n$$

where $z^2 = \sum_{j=1}^{n} z_j^2$ and $z \cdot x = \sum_{j=1}^{n} z_j x_j$.

Define the Bargmann transformation $T: \mathscr{S}'(R^n) \to \mathfrak{Z}^{+\infty}$ as follows:

$$Tf = \int_{R^n} f(x) A(z,x) dx, \quad f \in \mathscr{S}' \tag{1.3}$$

where the integration is formal, the real meaning is that the distribution f acts on $A(z, \cdot)$. In view of [8], T produces the following topological isomorphic:

$$\mathscr{S}' \to \mathfrak{Z}^{+\infty}, \quad \text{and} \quad L^2 \to \mathfrak{Z}^0 \quad \text{and} \quad \mathscr{S} \to \mathfrak{Z}^{-\infty} \tag{1.4}$$