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PERIODIC SOLUTIONS TO POROUS MEDIA EQUATIONS OF PARABOLIC-ELLIPTIC TYPE

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Abstract This paper is concerned with a equation, which is a model of filtration in partially saturated porous media, with mixed boundary condition of Dirichlet-Neumann type

$$\begin{cases} \partial_t b(u) - \nabla \cdot a[\nabla u + k(b(u))] = f & \text{in } (0,\infty) \times \Omega \\ u = h(t,x) & \text{on } (0,\infty) \times \Gamma_0 \\ \nu \cdot a[\nabla u + k(b(u))] = g(t,x) & \text{on } (0,\infty) \times \Gamma_1 \end{cases}$$

We have proved that there exists one and only one periodic solution of the problem under the data f, g and h with same period. Moreover, we have proved that the unique periodic solution ω is asymptotically statble in the sense that for any solution u of the problem

$$b(u(t)) - b(\omega(t)) \to 0 \quad \text{in } L^2(\Omega) \quad \text{as } t \to \infty.$$

Key Words Filtration equation; periodic solutions; asymptotic stability.
Classification 35K.

1. Introduction

This paper is concerned with a parabolic-elliptic equation of the following form

$$\partial_t b(u) - \nabla \cdot a[\nabla u + k(b(u))] = f \quad \text{in } (0,\infty) \times \Omega \qquad (1.1)$$

Here Ω is a bounded domain in $R^N (N \geq 1)$ with smooth boundary $\Gamma = \partial\Omega, b: R \to R, k: R \to R^N, f: (0,\infty) \times \Omega \to R$ are given functions and $a(x) = (a_{i,j}(x))$ is a positive definite symmetric matrix. This equation is a mathematical model of filtration in partially saturated porous media, where the saturation of flow is represented by $b(u)$ and the term $k(b(u))$ refers to the effect of gravitational force.

In this paper, equation (1.1) is studied with the following mixed boundary condition of Dirichlet-Neumann type

$$u = h(t,x) \qquad \text{on} \quad (0,\infty) \times \Gamma_0 \tag{1.2}$$
$$\nu \cdot a[\nabla u + k(b(u))] = g(t,x) \qquad \text{on} \quad (0,\infty) \times \Gamma_1 \tag{1.3}$$

where Γ_0 is a measurable subset of Γ with positive surface measure (i. e. $\text{meas}_\Gamma(\Gamma_0) > 0$), $\Gamma_1 = \Gamma \backslash \Gamma_0$, ν is the outward normal vector on Γ and h (resp. g) is a given function on $(0,\infty) \times \Gamma_0$ (resp. $(0,\infty) \times \Gamma_1$). This problem (1. 1)—(1. 3) with initial condition has been studied by many authors. For instance, see [1] for an existence-uniqueness result. Concerning the asymptotic behavior of solutions, we know (cf. [21]) that the stationary problem

$$\begin{cases} -\nabla \cdot a[\nabla u_\infty + k(b(u_\infty))] = f_\infty(x) & \text{in} \quad \Omega \\ u_\infty = h_\infty(x) \qquad \text{on} \quad \Gamma_0 \\ \nu \cdot a[\nabla u_\infty + k(b(u_\infty))] = g_\infty(x) & \text{on} \quad \Gamma_1 \end{cases} \tag{1.4}$$

has a unique solution u_∞ and any solution of (1. 1)—(1. 3) converges to u_∞ weakly in $H^1(\Omega)$ as $t \to \infty$, provided that $f(t,\cdot) \to f_\infty$, $g(t,\cdot) \to g_\infty$ and $h(t,\cdot) \to h_\infty$ in appropriate senses as $t \to \infty$.

In the present paper we are interested in periodic solutions to problem (1. 1)—(1. 3). We shall prove that there exists one and only one periodic solution of the problem under periodicity condition on the data f, g and h with same period. Moreover, we shall prove that the unique periodic solution ω is asymptotically stable in the sense that for any solution u of (1. 1)-(1. 3)

$$b(u(t)) - b(\omega(t)) \to 0 \quad \text{in } L^2(\Omega) \quad \text{as } t \to \infty$$

In proving the uniqueness of periodic solution, our main idea is to employ the same technique as the one in the proof of [8; Theorem 10. 7] (see the proof of Lemma 4. 1). The existence and asymptotic stability of periodic solution will be shown by using an order property for solutions u_1, u_2 as below

$$|[b(u_1(t)) - b(u_2(t))]^+|_{L^1(\Omega)} \leq |[b(u_1(s)) - b(u_2(s))]^+|_{L^1(\Omega)} \qquad \text{for any } s, t \text{ with } s \leq t$$

This property is originally due to Bénilan [3].

In the case $k = 0$, the periodic stability has been studied in [13], when the boundary condition is of mixed-type and of Signorini type on time-dependent parts of the boundary. The method employed in the present paper can be applicable, with the help of some order property of boundary flux (see [13; Proposition 4. 1]), to the boundary condition treated in [13].

For papers dealing with some related topics, see [2, 4, 6, 7, 9-12, 14-16, 20]. In particular, we refer to [5] for periodic behavior of solutions of the evolutionary dam problem in a rectangular domain. In [17-19, 22], periodic solutions have been studied