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THE ARAKAWA JACOBIAN METHOD AND A FOURTH-ORDER ESSENTIALLY NONOSCILLATORY SCHEME FOR THE BETA-PLANE BAROTROPIC EQUATIONS

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Abstract. In this paper we use the Arakawa Jacobian method [1] and the fourth-order essentially non-oscillatory (ENO-4) scheme of Osher and Shu [15] to solve the equatorial beta-plane barotropic equations. The Arakawa Jacobian scheme is a second order centred finite differences scheme that conserves energy and enstrophy. The fourth-order essentially non-oscillatory scheme is designed for Hamilton-Jacobi equations and traditionally used to track sharp fronts. We are interested in the performance of these two methods on the baratropic equations and determine whether they are adequate for studying the barotropic instability. The two methods are tested and compared on two typical exact solutions, a smooth Rossby wave-packet and a discontinuous shear, on the long-climate scale of 100 days. The numerical results indicate that the Arakawa Jacobian method conserves energy and enstrophy nearly exactly, as expected, captures the phase speed the Rossby wave, and achieves an overall second order accuracy, in both cases. The same properties are preserved by the ENO-4 scheme but the fourth order accuracy is observed only for the smooth Rossby wave solution while in the case of the discontinuous shear, it yields an overall third order accuracy, even in the smooth regions, away from the discontinuity.

Key words. Arakawa Jacobian, Essentially non-oscillatory schemes, Spectral methods, Finite difference, Large scale equatorial waves, Atmospheric circulation, Barotropic flow, Vorticity, Stream function.

1. INTRODUCTION

One of the important strategies for understanding atmospheric general circulation is to study the numerical solutions of its governing equations. The equatorial beta-plane barotropic equations, a simple atmospheric model, have been studied for more than half a century and are at the heart of a hierarchy of more complex models. The first successful numerical weather prediction model, used by Charny *et al* in 1950 [3], was based on the barotropic vorticity equation (BVE). A barotropic atmosphere is a single-layered fluid; under this assumption there is no vertical component, and hence the equation to be solved is two dimensional (2D). For theoretical investigation of the evolution of vortices, atmospheric researchers are still using the barotropic assumption. For example, the BVE is useful for modelling the movement of tropical cyclones [2]. The barotropic assumption is also used to model global wave patterns in the middle troposphere [19]. To model tropical cyclones, the computational domain is a midlatitude β-plane. The β-plane approximation is a linear approximation to the Coriolis parameter found by Taylor expansion [10] for small displacement in latitude. Scale analysis show that the nonlinear term is negligible.

Most numerical models of the BVE use finite differences or spectral methods. A recent state of the art method from the applied mathematics [13] to the problem of

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tropical climate modelling [12] showed that a non-oscillatory central scheme can accurately model equatorial waves without undue dissipation of energy but seems to suffer some serious shortcoming [12, 4] (see conclusion section). However, the Arakawa Jacobian scheme [1], which is specifically designed for the incompressible BVE, is widely used in the atmosphere-ocean community. The Arakawa Jacobian has the useful feature that both domain integrated enstrophy and domain integrated kinetic energy are conserved. It also conserves mean wavenumber; this prevents nonlinear instabilities from occurring. The third method which we can adapt to solve the incompressible BVE is the high-order essentially non-oscillatory scheme (ENO) of Osher and Shu [15]. The ENO scheme is a high order accurate finite difference scheme designed for problems with piecewise smooth solutions containing discontinuities. ENO schemes are traditionally used for hyperbolic conservation laws and Hamilton-Jacobi equations [15]. The key idea lies at the approximation level, where a nonlinear adaptive procedure is used to automatically choose the locally smoothest stencil, hence avoiding crossing discontinuities in the interpolation procedure as much as possible. ENO schemes have been quite successful in applications, especially for problems containing both shocks and complicated smooth solution structures, such as compressible turbulence simulations and aeroacoustic. The paper is organized as follows. In section 2, we present the barotropic equations on the equatorial β-plane. In Sections 3 and 4, we study the numerical methods needed for solving the equatorial beta-plane barotropic equations. The Arakawa Jacobian is used together with the second-order numerical solution of the Poisson equation, used to enforce the incompressibility constraint. The fourth-order essentially non-oscillatory (ENO-4) scheme is coupled with a fourth-order Poisson solver. We validate the numerical methods in Section 5, and a summary with conclusion is presented in Section 6.

2. The Barotropic Equations on an Equatorial β-plane

In standard nondimensional units, that are defined below, the barotropic equatorial β-plane equations, for the horizontal velocity, $\mathbf{v}$, and pressure, p, are given by

$$(1) \qquad \begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + y\mathbf{v}^{\perp} + \bigtriangledown p = 0, \\ \qquad \operatorname{div}\mathbf{v} \ = 0. \end{cases}$$

In (1), $\mathbf{v} = (u, v)$ with u, v are respectively the zonal (east-west) and meridional (north-south) velocity components. The operator $\nabla = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right)$ is the horizontal gradient vector and $\operatorname{div}\mathbf{v} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}$ is the horizontal divergence while the term $y\mathbf{v}^{\perp} = y\,(-v, u)$ represents the horizontal components of the Coriolis force due to the vertical component of Earth's rotation (beta effect). The nonlinear equations for the barotropic mode in (1) is derived from the full 3d geophysical flow equations by assuming a rigid lid and flat bottom, constant density and hydrostatic balance. These assumptions are sufficient to neglect the vertical velocity and viscosity in the rotating Boussinesq equations [5, 14].

The equations in (1) were nondimensionalized by using the characteristic units of equatorial synoptic scale dynamics [5, 14], so that the Coriolis gradient at the equator is normalized to $\beta = 1$: the velocity scale is the gravity wave speed $c =$