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UNIQUENESS OF NON-FLAT SOLUTION FOR A CLASS OF FUCHSIAN EQUATIONS WITH COMPLEX COEFFICIENTS*

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Abstract

In this paper, we have studied the uniqueness of C^k-flat solution C_f^k for the first and second order Fuchsian equations in non-C^∞ structure. We have obtained several sufficient conditions, which show that K is dependent not only on the principle parts of the operator but also on the singular coefficients of the lower order terms.

When the initial surface is the singular hypersurface of an equation, the Cauchy problem of a Fuchsian equation is very complicated and the uniqueness is more difficult. Up to now, the problem was studied only in the range of the flat solution (see [1] and its references), or that can be changed into a flat solution[2]. Nevertheless, for a singular equation, the uniqueness of the Cauchy problem is not equivalent to the uniqueness of flat solution[3]. We found the following result in [2]: if the Fuchsian roots $\mu_j(x)$ satisfy $\mu_j(0) \overline{\in} \{h, h+1, \ldots\ldots\}$, then there exists a small neighborhood θ of the origin such that the $C^\infty(\theta)$ solution u of the Fuchsian equation is unique if it satisfies $\partial_t^j u(x,0)=0$, $0 \leq j \leq h-1$. But two questions have arisen: (1) how about the non-$C^\infty(\theta)$ solution? For example, the uniqueness of non-regular solution; (2) does there exist h which satisfies the above condition and that C_f^k solution is not unique if we don't confine the Fuchsian roots? As an example, we consider the operator $\partial_t + a(x)t^{-1} + \sum a_j(x,t)\partial_{x_j}$, where $a =$ constant < 0. $u = t^{-a}$ is a non-zero solution of the homogeneous Cauchy problem. Let $a = -3/2$, then $\mu(0) = \frac{3}{2}$, $h = 2$, in [2], but $u = t^{3/2}$ is indeed a non-zero solution which satisfies $\partial_t^j u(x,0) = 0$, $0 \leq j \leq 1$.

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In this paper, we study the equation more carefully in non- C^∞ and non-flat solution framework. We introduce a kind of singular transformation and a kind of mixed transformation, set up corresponding estimates and obtain some sufficient conditions about the uniqueness of C_f^k solution.

(1)

Let $\Omega \subset R^n$ be an open subset, and denote point coordinates by $x = (x_1, \ldots, x_n)$. In $(0, T_0) \times \Omega$, we consider

$$L_0 \equiv t\partial_t + a(x) + t\sum_{j=1}^{n} a_j(x, t)\partial_{x_j} + tb(x, t) \tag{1}$$

where $t \in (0, T_0)$, $T_0 > 0$, a_j and b are C^1-complex functions, $a(x)$ is a real functions.

Definition. *$u(x, t)$ is called the C^k flat solution of an equation $Lu = 0$ (denoted by $u \in C_f^k$) if it satisfies $Lu = 0$ and $\partial_t^j u(x, 0) = 0$, $(0 \le j \le k)$, where L is a linear partial differential operator of order m . If L is hyperbolic, $k = m - 1$, then a C_f^k flat solution is the solution of the homogeneous Cauchy problem. If $k = \infty$, it is a flat solution.*

From $C_f^k \subset C_f^{k_0}$, where $k \ge k_0$, we have

Proposition 1. *If C_f^k solution of $Lu = 0$ is unique, then the $C_f^{k_0}$ solution of $Lu = 0$ is unique if $k \ge k_0$.*

We introduce a singular transformation

$$x = x, \quad t = (\delta - |x|^2)T \tag{$*$}$$

Then (1) becomes

$$\widetilde{L}_1 \equiv AT\partial_T + a(x) + (\delta - |x|^2)T\sum_{j=1}^{n}\widetilde{a}_j(x, T)\partial_{x_j} + (\delta - |x|^2)T\widetilde{b}(x, T) \tag{2}$$

where $A = 1 + 2T\sum_j x_j \widetilde{a}_j$, $\widetilde{a}_j(x, T) = a_j(x, (\delta - |x|^2)T)$, with $\widetilde{b}(x, T)$ similarly defined. Thus $A(0, 0) = 1$. Let T_0, δ be small enough such that $|A(x, T)| \ge C_0 > 0$, if $|x|^2 \le \delta$ and $0 \le T \le T_0$. Dividing both sides of (2) by AT and replacing T by t , we have

$$L_1 u = \partial_t u + a_1(x, t)t^{-1}u + \sum \widetilde{a}_j \frac{(\delta - |x|^2)}{A}\partial_{x_j} u + \frac{(\delta - |x|^2)}{A}\widetilde{b}(x, t)u = 0 \tag{2$'$}$$

where $a_1 = \dfrac{a(x)}{A} = a(x) - \dfrac{2a(x)t\sum \widetilde{a}_j x_j}{A}$. Putting the second term in a_1 into the last of $L_1 u$, we obtain

$$Lu = \partial_t u + a(x)t^{-1}u + \sum a_j' \partial_{x_j} u + b'(x, t)u = 0 \tag{3}$$

It is easy to see that $|a_j'|$ and $|b'|$ can be small enough if $|x|^2 \le \delta$, $0 \le T \le T_0$, and