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A Fitted Numerov Method for Singularly Perturbed Parabolic Partial Differential Equation with a Small Negative Shift Arising in Control Theory

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Abstract. In this paper, a fitted Numerov method is constructed for a class of singularly perturbed one-dimensional parabolic partial differential equations with a small negative shift in the temporal variable. Similar boundary value problems are associated with a furnace used to process a metal sheet in control theory. Here, the study focuses on the effect of shift on the boundary layer behavior of the solution via finite difference approach. When the shift parameter is smaller than the perturbation parameter, the shifted term is expanded in Taylor series and an exponentially fitted tridiagonal finite difference scheme is developed. The proposed finite difference scheme is unconditionally stable. When the shift parameter is larger than the perturbation parameter, a special type of mesh is used for the temporal variable so that the shift lies on the nodal points and an exponentially fitted scheme is developed. This scheme is also unconditionally stable. The applicability of the proposed methods is demonstrated by means of two examples.

AMS subject classifications: 65L11

Key words: Singular perturbations, parabolic partial differential equation, exponentially fitted method, differential-difference equations.

1. Introduction

Singularly perturbed partial differential equations are the equations in which the unknown function and its derivatives are evaluated at the same instance while in a singularly perturbed delay partial differential equation the past history is also taken into consideration while evaluating the unknown function and its derivatives. Such model problems occur from the modeling of biological, chemical, and physical systems which are characterized by both spatial and temporal variables and exhibit various spatio-temporal pat-

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terns [1–5]. The mathematical model relating an automatically controlled furnace to process metal sheets [1] is given by the equation:

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + v(g(u(x,t-\tau)))\frac{\partial u(x,t)}{\partial x} + c[f(u(x,t-\tau)) - u(x,t)].$$

This equation is defined on a one-dimensional spatial domain $0 < x < 1$, where v is the instantaneous material strip velocity depending on a prescribed spatial average of the time-delayed temperature distribution $u(x,t-\tau)$, and f represents a distributed temperature source function depending on $u(x,t-\tau)$. The material strip to be heated is fed into the furnace by rollers whose speed is regulated by a speed controller. The furnace temperature is varied by means of a heater actuated by a heater controller. The control objective is to maintain a desired spatial temperature distribution in the incoming material within the furnace. This may be accomplished by placing temperature transducers along the material strip. A computer uses the information from the transducers to generate the appropriate control signals for the heater and feed-roller speed controllers. Owing to the possible presence of time delays in actuation, and in information transmission and processing, the controlled signals may be delayed in time.

Extensive literature has been developed over the last two decades on the singularly perturbed partial differential equations [6–17], but the theory and numerical solutions of singularly perturbed delay partial differential equations are still at the initial stage. Ansari et al. [18] in their work considered a Dirichlet boundary value problem of singularly perturbed delay parabolic partial differential equation. A numerical method comprising a standard finite difference operator on a rectangular piecewise uniform fitted mesh of $N_x \times N_t$ elements condensing in the boundary layers is developed. The method is proved to be robust with respect to the small parameter. Yulan Wang [19] considered a similar type of singularly perturbed delay parabolic partial differential equation wherein the domain is divided into three sub-domains namely the two inner regions and an outer region and a reliable analytical technique is developed. Bashier and Patidar [20] developed a robust fitted operator finite difference method for the numerical solution of a singularly perturbed delay parabolic partial differential equation. Sufficient analysis is carried out to verify the validity of the solutions obtained.

In this paper, we presented exponentially fitted finite difference methods for a class of singularly perturbed one-dimensional parabolic partial differential equations with a small negative shift in the temporal variable. Briefly, the outline is as follows: In Section 2, we state the problem. In Section 2.2, the finite difference method is developed considering the shift parameter to be smaller than the perturbation parameter, the truncation error in the finite difference scheme is calculated and stability analysis is carried out. In Section 2.4, the shift parameter is considered to be larger than the perturbation parameter and the finite difference method is developed. The truncation error is calculated and stability analysis is carried out. To demonstrate the efficiency of the proposed methods, numerical experiments are carried out for two test problems and the results are given in Section 3. Finally the conclusions are given in the last section.