Numer. Math. Theor. Meth. Appl. Vol. 7, No. 1, pp. 107-122
doi: 10.4208/nmtma.2014.1236nm February 2014

A Study of Multiple Solutions for the Navier-Stokes Equations by a Finite Element Method

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Received 8 November 2012; Accepted (in revised version) 28 April 2013

Available online 24 January 2014

Abstract. In this paper, a finite element method is proposed to investigate multiple solutions of the Navier-Stokes equations for an unsteady, laminar, incompressible flow in a porous expanding channel. Dual or triple solutions for the fixed values of the wall suction Reynolds number R and the expansion ratio α are obtained numerically. The computed multiple solutions for the symmetric flow are validated by comparing them with approximate analytic solutions obtained by the similarity transformation and homotopy analysis method. Unlike previous works, our method deals with the Navier-Stokes equations directly and thus has no similarity and other restrictions as in previous works. Finally we use the method to study multiple solutions for three cases of the asymmetric flow (which has not been studied before using the similarity-type techniques).

AMS subject classifications: 76M10, 76D05, 74H15

Key words: Finite element method, Navier-Stokes equations, porous channel, expanding walls.

1. Introduction

In recent decades, there are growing interests in studying the laminar flow in channels or pipes with porous and expanding or contracting walls due to their relevance to a number of biological and engineering models, such as the transport of biological fluids through contracting or expanding vessels, the synchronous pulsation of porous diaphragms, the modeling of air circulation in the respiratory system and the model of the regression of the burning surface in solid rocket motors. Furthermore, the classical Berman's problem can be regarded as a special case of this model when the walls are stationary.

In order to study the transpiration cooling, Berman [1] established the model to describe the diffusion of fluids at a porous channel. He regarded Reynolds number as a small

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parameter and obtained the asymptotic solution for the first time. Since then a large number of analytic and numerical investigations for the problem have been done. In general, most of the researchers reduce the Navier-Stokes equations to a boundary value problem of a 4th-order nonlinear ordinary differential equation (ODE) through a similarity transformation and then obtain its asymptotic or numerical solutions. For example, Yuan [2], Terrill and Shrestha [3], Suryaprakashrao [4] obtained asymptotic solutions using perturbation method and discussed the relation between the velocity field and the Reynolds number. In the numerical investigation of the solutions of the flow (ODE) in a stationary channel, they use an initial value method to solve the boundary value problem. The shooting method combined with a Runge-Kutta integrator was mainly employed. Terrill [3,5,6] may be the earliest to have numerical studies for the laminar flow of different Reynolds number R in a porous channel and obtained one solution for a few Reynolds numbers R. Robinson [7], Lu et al. [8] and Zaturska et al. [9] discussed the multiplicity of solution for the flow of different R in a porous channel with stationary walls by numerically solving the nonlinear ODE. All of numerical studies for the similarity-transformed ODE model with stationary walls have revealed that one solution exists for $-12.165 \leq R \leq 0$ and three solutions exist for$-\infty < R < -12.165$.

However, very little has been done for the multiple solutions of the flow in a porous channel with moving walls. Majdalani and Zhou [10–13], Asghar et al. [14] and Saeed et al. [15] discussed the flow in a deforming channel using perturbation method, Adomian decomposition method (AMD) and homotopy analysis method respectively, but they did not focus on the multiplicity of the solution. Dauenhauer and Majdalani [16] believed that multiple solutions should exist for the flow in a porous channel with expanding or contracting walls and should be influenced by both R and expansion ratio α. Recently, Xu et al. [17] investigated the multiple solutions of the flow in a porous channel with moving walls using the homotopy analysis method. They obtained two new profiles that are complementary to the solutions explored by Dauenhauer and Majdalani [16].

Durlofsky and Brady [18] indicated that similarity solutions are important in helping us understand the behavior of fluids. However, there is no assurance that these solutions represent a physically realizable flow and that these solutions are physically stable. This is the motivation that we study the multiple solutions by solving the original governing Navier-Stokes equations without making a similarity transformation. Furthermore, our method based directly on the original equations may be applied to general problems without any restriction accompanied with the similarity transformation (for example, the assumption of constant expansion ratio is not necessary in our method).

In this paper, we shall directly solve the Navier-Stokes equations using the finite element method, which is a very popular numerical method for partial differential equations, especially for various fluid flow problems (see e.g., [19–21]). We aim to study the multiple solutions for the flow in a porous channel with moving walls by employing a continuous finite element method. The moving walls are converted to fixed walls through a simple variable transformation. Since we deal with the time dependent governing equations directly the dynamic stability of these solutions may be justified through the solving process. In Section 2, we introduce the model of the laminar flow in a porous channel with expand-