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Multigrid Solution of a Lavrentiev-Regularized State-Constrained Parabolic Control Problem

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Abstract. A mesh-independent, robust, and accurate multigrid scheme to solve a linear state-constrained parabolic optimal control problem is presented. We first consider a Lavrentiev regularization of the state-constrained optimization problem. Then, a multigrid scheme is designed for the numerical solution of the regularized optimality system. Central to this scheme is the construction of an iterative pointwise smoother which can be formulated as a local semismooth Newton iteration. Results of numerical experiments and theoretical twogrid local Fourier analysis estimates demonstrate that the proposed scheme is able to solve parabolic state-constrained optimality systems with textbook multigrid efficiency.

AMS subject classifications: 35K10, 49K20, 49J20, 49M05, 65M55, 65C20

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1. Introduction

Optimal control of parabolic systems occurs in many application fields such as chemical reaction simulations and biomedical sciences, among other important fields [3, 4, 20]. These problems require the development of algorithms that are fast and robust with respect to the optimization parameters. Recent developments [1–3, 10] show that a successful framework to develop such algorithms is represented by space-time collective-smoothing multigrid schemes. In fact, Fourier analysis estimates [6, 10] and results of numerical experiments with linear [1] and nonlinear [3, 4, 10] parabolic control problems demonstrate that space-time multigrid schemes provide optimal control solutions with mesh-independent convergence and robustness with respect to the value of the control parameters.

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Previous contributions to the multigrid solution of parabolic control problems [1, 4, 6, 8, 12] have focused on first-order time discretization, while higher-order space-time discretization and constraints on the control have been considered in [10]. In this contribution, the authors found that the Crank-Nicolson scheme is not a convenient choice while multistep backward differencing schemes are advantageous in the design of very efficient pointwise and linewise smoothers.

In this paper, we contribute to the field of space-time multigrid methods for the case of parabolic optimal control problems with state-constraints and second-order space-time discretization. Our multigrid approach is formulated based on criteria proposed in [1, 2, 10] combining space-time collective smoothing multigrid schemes and Lavrentiev regularization [21,25]. In this framework, the smoothing procedures are pointwise iterative schemes that update the optimization variables collectively and use projection to satisfy the inequality constraints. We show that these iterative schemes can be interpreted as local semismooth Newton methods [13,23,24] applied to the regularized state-constrained problems.

In the next section, we formulate a state-constrained linear parabolic optimal control problem. Further, we obtain a Lavrentiev regularization of the problem and characterize the optimal solutions as solutions of the corresponding regularized optimality system. In Section 3, we discuss a space-time second-order discretization of the optimality system. In Section 4, we illustrate the space-time multigrid framework and focus on the construction of an efficient pointwise smoother. In Section 5, we investigate the proposed smoother using results of twogrid local Fourier analysis to discuss the convergence properties of the multigrid scheme with the pointwise smoother. We obtain smoothing-factor and multigrid convergence-factor estimates that predict typical textbook multigrid efficiency and robustness with respect to the values of the control parameters. Further, we present novel insight that shows that the resulting smoothers can be interpreted as local semismooth Newton schemes. In Section 6, detailed numerical experiments are carried out. The numerical results demonstrate the ability of the proposed multigrid framework to provide efficient solutions to state constrained linear parabolic optimal control problems. Besides, we discuss the application of the receding-horizon methodology [3, 14] to achieve long-time tracking of a desired trajectory with state constraints. A section of conclusion completes this work.

2. A state-constrained parabolic optimal control problem

Optimal control problems are defined for the purpose of determining the optimal way to influence dynamical systems towards a given task. Our optimal control problem consists of a parabolic governing system, a distributed control mechanism, and a criterion defining the cost functional, that models the purpose of the control and describes the cost of its action. The formulation of an optimal control problem is then to minimize the cost functional under the constraint given by the modeling equations. The solution to this problem is characterized by first-order optimality conditions given by the optimality system. In particular, we focus on state-constrained parabolic optimal control problems where the configuration of the controlled system is subject to functional constraints. For a more general and detailed discussion on optimal control problems see, e.g., [17,25].