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Discrete Maximum Principle and a Delaunay-Type Mesh Condition for Linear Finite Element Approximations of Two-Dimensional Anisotropic Diffusion Problems

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Abstract. A Delaunay-type mesh condition is developed for a linear finite element approximation of two-dimensional anisotropic diffusion problems to satisfy a discrete maximum principle. The condition is weaker than the existing anisotropic non-obtuse angle condition and reduces to the well known Delaunay condition for the special case with the identity diffusion matrix. Numerical results are presented to verify the theoretical findings.

AMS subject classifications: 65N30, 65N50

Key words: Anisotropic diffusion, discrete maximum principle, finite element, mesh generation, Delaunay triangulation, Delaunay condition.

1. Introduction

We are concerned with the linear finite element (FEM) solution of the two-dimensional anisotropic diffusion equation

$$-\nabla \cdot (\mathbb{D}\,\nabla u) = f, \qquad \text{in} \quad \Omega \tag{1.1}$$

subject to the Dirichlet boundary condition

$$u = g, \qquad \text{on} \quad \partial\Omega, \tag{1.2}$$

where $\Omega \in \mathbb{R}^2$ is a connected polygonal domain, f and g are given functions, and $\mathbb{D} = \mathbb{D}(x,y)$ is the diffusion matrix assumed to be symmetric and strictly positive definite on Ω. This boundary value problem (BVP) is a model of anisotropic diffusion problems arising in various fields such as plasma physics [15–17, 34, 36, 38], petroleum reservoir simulation

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[1,2,10,13,32], and image processing [6,7,21,33,35,43]. A distinct feature of the BVP is that its solution satisfies the maximum principle and is monotone when

$$f(x,y) \le 0, \qquad \text{for all } (x,y) \in \Omega.$$

A challenge in the numerical solution of the BVP is to design a scheme so that the resulting numerical approximations satisfy a discrete maximum principle (DMP).

Development of DMP satisfaction schemes for solving diffusion problems has attracted considerable interest in the past; e.g., see [4,5,8,9,18–20,23,27,39–42,44] for isotropic diffusion problems where

$$\mathbb{D} = a(x,y)I$$

with $a(x,y)$ being a scalar function and [10,11,15–17,22,24–26,28–32,36] for anisotropic diffusion problems where $\mathbb{D}(x,y)$ can be heterogeneous and anisotropic. For example, Ciarlet and Raviart [9] (also see Brandts *et al.* [4]) show that the linear finite element method for an isotropic diffusion problem (in any dimension) satisfies DMP when the mesh is simplicial and satisfies the non-obtuse angle condition which requires that the dihedral angles of mesh elements be non-obtuse. In two dimensions and for the special case $\mathbb{D} = I$, the condition can be replaced by the Delaunay condition, a weaker condition which only requires the sum of any pair of angles opposite a common edge to be less than or equal to π [27,41]. Moreover, Xu and Zikatanov [44] show that the non-obtuse angle condition at edges where the diffusion coefficient is discontinuous and the Delaunay condition at other places guarantee DMP satisfaction.

Note that the Delaunay condition may be insufficient for DMP satisfaction in three dimensions since it is insufficient to guarantee the M-matrix property of the stiffness matrix in 3D (see Letniowski [27]), a crucial property used in most of DMP satisfaction proofs. Recently, Li and Huang [28] generalize the non-obtuse angle condition to anisotropic diffusion problems and obtain a so-called anisotropic non-obtuse angle condition which requires the dihedral angles of mesh elements, when measured in a metric depending on $\mathbb{D}$, to be non-obtuse. It is thus natural to ask if the anisotropic non-obtuse angle condition can be replaced by a weaker, Delaunay-type mesh condition for anisotropic diffusion problems in 2D.

The objective of this paper is to extend the Delaunay condition to anisotropic diffusion problems. A Delaunay-type mesh condition is developed for the DMP satisfaction of linear finite element approximations for those problems. It is shown that the new condition reduces to the Delaunay condition for the special case $\mathbb{D} = I$ and is weaker than the anisotropic non-obtuse angle condition developed in [28]. We arrive at the new condition by investigating the stiffness matrix as a whole. This is different from [28] where only local stiffness matrices on individual elements are considered. The main result is given in Theorem 4.1.

This paper is organized as follows. The linear finite element formulation for BVP (1.1) and (1.2) is given in Section 2. Section 3 is devoted to the description and the geometric interpretation of the anisotropic non-obtuse angle condition. The Delaunay-type mesh condition is developed in Section 4 while illustrative numerical results are presented in Section 5. Finally, Section 6 contains conclusions and comments.