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Stability Analysis of Runge-Kutta Methods for Nonlinear Neutral Volterra Delay-Integro-Differential Equations

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Abstract. This paper is concerned with the numerical stability of implicit Runge-Kutta methods for nonlinear neutral Volterra delay-integro-differential equations with constant delay. Using a Halanay inequality generalized by Liz and Trofimchuk, we give two sufficient conditions for the stability of the true solution to this class of equations. Runge-Kutta methods with compound quadrature rule are considered. Nonlinear stability conditions for the proposed methods are derived. As an illustration of the application of these investigations, the asymptotic stability of the presented methods for Volterra delay-integro-differential equations are proved under some weaker conditions than those in the literature. An extension of the stability results to such equations with weakly singular kernel is also discussed.

AMS subject classifications: 65L05, 65L06, 65L20, 34K40

Key words: Neutral differential equations, Volterra delay-integro-differential equations, Runge-Kutta methods, stability.

1. Introduction

Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote a given inner product and the corresponding induced norm in the complex N-dimensional space $\mathbf{C}^N$. In this paper we consider the stability of Runge-Kutta methods (RKMs) for nonlinear neutral Volterra delay-integro-differential equations (NVDIDEs) with constant delay $\tau > 0$,

$$y'(t) = f\left(t, y(t), y(t-\tau), \int_{t-\tau}^{t} K(t,\theta,y(\theta),y'(\theta))d\theta\right), \quad t \geq 0, \tag{1.1}$$

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subject to

$$y(t) = \phi(t), \quad t \in [-\tau, 0], \tag{1.2}$$

where $f : [0,\infty) \times \mathbf{C}^N \times \mathbf{C}^N \times \mathbf{C}^N \to \mathbf{C}^N$ and $K : [0,\infty) \times [-\tau,\infty) \times \mathbf{C}^N \times \mathbf{C}^N \to \mathbf{C}^N$ are continuous functions, ϕ is a given C^1-function.

As special cases of Eq. (1.1), we have delay differential equations (DDEs)

$$y'(t) = f(t, y(t), y(t-\tau)), \quad t \geq 0, \tag{1.3}$$

and Volterra delay-integro-differential equations (VDIDEs)

$$y'(t) = f\left(t, y(t), y(t-\tau), \int_{t-\tau}^{t} K(t,\theta,y(\theta))d\theta\right), \quad t \geq 0. \tag{1.4}$$

Delay differential equations with constant delays have been investigated extensively in the past. For the literature concerned the stability of the true solution and the numerical solution to DDEs, we refer the reader to [19, 27, 34], and the references in [5, 24, 46]. Numerical methods for solving VDIDEs have been also studied by many authors (see [3, 10, 25] and references therein). Using the generalized Halanay inequality proved by Baker and Tang in [2], Zhang and Vandewalle [43] obtained the stability results for the true solution to

$$y'(t) = f\left(t, y(t), G\left(t, y(t-\tau), \int_{t-\tau}^{t} K(t,\theta,y(\theta))d\theta\right)\right), \quad t \geq 0. \tag{1.5}$$

They also investigated the stability of the numerical solution of a discretized form of (1.5). In [44, 45], they further considered the nonlinear stability of RKMs and general linear methods (GLMs) for VDIDEs (1.4), respectively.

There is a growing interest in developing numerical methods for solving NVDIDEs. This class of equations arises in many applications (see [10, 23, 39] and references therein) and often occurs in two forms: the general nonlinear delay IDEs of neutral type

$$y'(t) = f\left(t, y(t), y(t-\tau(t)), y'(t-\tau(t)), \int_{t-\tau(t)}^{t} K(t,\theta,y(\theta),y'(\theta))d\theta\right), \quad t \geq 0, \tag{1.6}$$

and the neutral equations of the "Hale's form"

$$\frac{d}{dt}\left[y(t) - \int_{t-\tau(t)}^{t} K(t,\theta,y(\theta))d\theta\right] = f\left(t, y(t), y(t-\tau(t)), y'(t-\tau(t))\right), \quad t \geq 0, \tag{1.7}$$

where $\tau(t) \leq t$ is a sufficiently smooth function. In [10] (see also [7, 13]), Brunner systematically discussed the existence and uniqueness of the solution to these two forms of equations and the convergence of collocation methods for them. Note that as far back as the 1980's, Jackiewicz gave the convergence results of Adams methods [20] and quasi-linear multi-step methods and variable step predictor-corrector methods [21] for solving