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Cubature Formula and Interpolation on the Cubic Domain

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Abstract. Several cubature formulas on the cubic domains are derived using the discrete Fourier analysis associated with lattice tiling, as developed in [10]. The main results consist of a new derivation of the Gaussian type cubature for the product Chebyshev weight functions and associated interpolation polynomials on $[-1,1]^2$, as well as new results on $[-1,1]^3$. In particular, compact formulas for the fundamental interpolation polynomials are derived, based on $n^3/4+\mathcal{O}(n^2)$ nodes of a cubature formula on $[-1,1]^3$.

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1. Introduction

For a given weight function W supported on a set $\Omega \in \mathbb{R}^d$, a cubature formula of degree $2n-1$ is a finite sum, $L_n f$, that provides an approximation to the integral and preserves polynomials of degree up to $2n-1$; that is,

$$\int_\Omega f(x)W(x)dx = \sum_{k=1}^{N} \lambda_k f(x_k) =: L_n f, \quad \forall f \in \Pi^d_{2n-1},$$

where Π^d_n denotes the space of polynomials of total degree at most n in d variables. The points $x_k \in \mathbb{R}^d$ are called *nodes* and the numbers $\lambda_k \in \mathbb{R} \setminus \{0\}$ are called *weights* of the cubature.

Our primary interests are Gaussian type cubature, which has minimal or nearer minimal number of nodes. For $d=1$, it is well known that Gaussian quadrature of degree $2n-1$ needs merely $N=n$ nodes and these nodes are precisely the zeros of the orthogonal polynomial of degree n with respect to W. The situation for $d>1$, however, is much more

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complicated and not well understood in general. As in the case of $d=1$ for which a cubature of degree $2n-1$ needs at least n nodes, the cubature of degree $2n-1$ for $d \ge 1$ needs $N \ge \dim \Pi^d_{n-1}$ number of nodes, but few formulas are known to attain this lower bound (see, e.g., [1, 10]). In fact, for the centrally symmetric weight function (symmetric with respect to the origin), it is known that the number of nodes, N, of a cubature of degree $2n-1$ in two dimension satisfies the lower bound

$$N \ge \dim \Pi^2_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor, \tag{1.1}$$

known as Möller's lower bound [11]. It is also known that the nodes of a cubature that attains the lower bound (1.1), if it exists, are necessarily the common zeros of $n+1-\lfloor \frac{n}{2} \rfloor$ orthogonal polynomials of degree n with respect to W. Similar statements on the nodes hold for cubature formulas that have number of nodes slightly above Möller's lower bound, which we shall call cubature of *Gaussian type*. These definitions also hold in d-dimension, where the lower bound for the number of nodes for the centrally symmetric weight function is given in [12].

There are, however, only a few examples of such formulas that are explicitly constructed and fewer still can be useful for practical computation. The best known example is $\Omega = [-1,1]^d$ with the weight function

$$W_0(x) := \prod_{i=1}^{d} \frac{1}{\sqrt{1-x_i^2}} \qquad \text{or} \qquad W_1(x) := \prod_{i=1}^{d} \sqrt{1-x_i^2} \tag{1.2}$$

and only when $d=2$. In this case, several families of Gaussian type cubature are explicitly known, they were constructed ([13, 17]) by studying the common zeros of corresponding orthogonal polynomials, which are product Chebyshev polynomials of the first kind and the second kind, respectively. Furthermore, interpolation polynomial bases on the nodes of these cubature formulas turn out to possess several desirable features ([18], and also [5]). On the other hand, studying common zeros of orthogonal polynomials of several variables is in general notoriously difficult. In the case of (1.2), the product Chebyshev polynomials have the simplest structure among all orthogonal polynomials, which permits us to study their common zeros and construct cubature formulas in the case $d=2$, but not yet for the case $d=3$ or higher.

The purpose of the present paper is to provide a completely different method for constructing cubature formulas with respect to W_0 and W_1. It uses the discrete Fourier analysis associated with lattice tiling, developed recently in [10]. This method has been used in [10] to establish cubature for *trigonometric functions* on the regular hexagon and triangle in $\mathbb{R}^2$, a topic that has been studied in [15, 16], and on the rhombic dodecahedron and tetrahedron of $\mathbb{R}^3$ in [9]. The cubature on the hexagon can be transformed, by symmetry, to a cubature on the equilateral triangle that generates the hexagon by reflection, which can in turn be further transformed, by a nontrivial change of variables, to Gaussian cubature formula for algebraic polynomials on the domain bounded by Steiner's hypercycloid. The theory developed in [10] uses two lattices, one determines the domain of integral and