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Uniform Convergence Analysis for Singularly Perturbed Elliptic Problems with Parabolic Layers

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Abstract. In this paper, using Lin's integral identity technique, we prove the optimal uniform convergence $\mathcal{O}(N_x^{-2}\ln^2 N_x + N_y^{-2}\ln^2 N_y)$ in the L^2-norm for singularly perturbed problems with parabolic layers. The error estimate is achieved by bilinear finite elements on a Shishkin type mesh. Here N_x and N_y are the number of elements in the x- and y-directions, respectively. Numerical results are provided supporting our theoretical analysis.

AMS subject classifications: 65M60, 65M12, 65M15
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Dedicated to Professor Yucheng Su on the Occasion of His 80th Birthday

1. Introduction

It is well-known that singularly perturbed problems [14] appear in many application areas, and solving them efficiently is very challenging due to its inherent multiscales. Hence singularly perturbed problems often serve as challenging benchmark problems. In the past decade, many novel numerical methods for solving these problems have been proposed (e.g., [9, 11] and references cited therein).

In this paper, we consider the singularly perturbed elliptic problem:

$$-\varepsilon^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial u}{\partial y}+u=f(x,y) \quad \text{in} \ \ \Omega\equiv(0,1)^2\,, \tag{1.1}$$

$$u=0 \quad \text{on} \ \ \partial\Omega\,, \tag{1.2}$$

where $0<\varepsilon\ll 1$. This problem is different from most singularly perturbed problems in that it not only has the ordinary exponential boundary layer at $y=1$, but also has

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parabolic boundary layers at both $x = 0$ and $x = 1$. Such parabolic layer problems have been studied by many researchers (e.g., [1,3,10,12,15]). Li [3] has studied (1.1)-(1.2) using bilinear finite elements with Shishkin type meshes, but only first-order convergence in the L^2-norm was proved. Later, Roos *et al.* [10, 15] studied similar problems, and only first-order convergence in the energy norm was obtained for bilinear finite elements. However, we observed almost second-order convergence in the L^2-norm in our numerical tests [3]. In this paper, we fill the gap by providing a rigorous proof for this phenomenon. We like to remark that the proof is not trivial and needs to use the so-called Lin's integral identity technique developed in early 1990 by Lin and his group [5,6,8] for finite element superconvergence. More details can be found in a later book by Lin and Yan [7]. Li [2,4] was the first to use such technique to prove optimal uniform convergence for reaction-diffusion type singularly perturbed problems. Now it becomes an indispensable tool for proving optimal convergence for singularly perturbed problems [13,16].

The rest paper is organized as follows. In Section 2, we construct the Shishkin mesh based on the boundary layers of the analytic solution. Then we present our finite element method and the interpolation error in Section 3. In Section 4, we provide the optimal convergence error analysis. A numerical example is provided in Section 5 to support our theoretical analysis.

2. Solution decomposition and Shishkin mesh

To reduce technicality as Roos *et al.* [10, 15], we assume that the analytic solution of our problem (1.1)-(1.2) exists the following decomposition:

Theorem 2.1. *If $f \in C^{3,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$, satisfies the compatibility condition $f(0,0) = f(0,1) = f(1,0) = f(1,1) = 0$, then the solution of* (1.1)-(1.2) *can be decomposed as*

$$u = S + E_1 + E_2 + E_3, \tag{2.1}$$

where for all $(x,y) \in \overline{\Omega}$ and $0 \le i + j \le 3$,

$$\begin{aligned}
&\Big|\frac{\partial^{i+j} S}{\partial x^i \partial y^j}(x,y)\Big| \le C, \quad \Big|\frac{\partial^{i+j} E_1}{\partial x^i \partial y^j}(x,y)\Big| \le C\epsilon^{-2j} e^{-(1-y)/\epsilon^2}, \\
&\Big|\frac{\partial^{i+j} E_2}{\partial x^i \partial y^j}(x,y)\Big| \le C\epsilon^{-i}(e^{-x/\epsilon} + e^{-(1-x)/\epsilon}), \\
&\Big|\frac{\partial^{i+j} E_3}{\partial x^i \partial y^j}(x,y)\Big| \le C\epsilon^{-(i+2j)} e^{-(1-y)/\epsilon^2}(e^{-x/\epsilon} + e^{-(1-x)/\epsilon}).
\end{aligned}$$

To construct a Shishkin type mesh accordingly, we assume that the positive integers N_x and N_y are divisible by 4, where N_x and N_y denote the number of elements in the x- and y-directions, respectively. In the x-direction, we first divide the interval $[0,1]$ into the subintervals

$$[0, \sigma_x], \quad [\sigma_x, 1 - \sigma_x], \quad [1 - \sigma_x, 1]\ .$$