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A Finite Difference Scheme on *a Priori* Adapted Meshes for a Singularly Perturbed Parabolic Convection-Diffusion Equation

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Abstract. A boundary value problem is considered for a singularly perturbed parabolic convection-diffusion equation; we construct a finite difference scheme on *a priori* (sequentially) *adapted* meshes and study its convergence. The scheme on *a priori* adapted meshes is constructed using a majorant function for the singular component of the discrete solution, which allows us to find *a priori* a subdomain where the computed solution requires a further improvement. This subdomain is defined by the perturbation parameter ε, the step-size of a uniform mesh in x, and also by the required accuracy of the discrete solution and the prescribed number of refinement iterations K for improving the solution. To solve the discrete problems aimed at the improvement of the solution, we use uniform meshes on the subdomains. The error of the numerical solution depends weakly on the parameter ε. The scheme converges almost ε-uniformly, precisely, under the condition $N^{-1} = o(\varepsilon^{\nu})$, where N denotes the number of nodes in the spatial mesh, and the value $\nu = \nu(K)$ can be chosen arbitrarily small for suitable K.

AMS subject classifications: 65M06, 65M12, 65M15, 65M50

Key words: Singular perturbations, convection-diffusion problem, piecewise-uniform mesh, *a priori* adapted mesh, almost ε-uniform convergence.

Dedicated to Professor Yucheng Su on the Occasion of His 80th Birthday

1. Introduction

At present, there are fairly well-developed methods for constructing ε-uniformly convergent schemes on meshes that are *a priori* adapted in a boundary layer region and not changing in the computational process, or, in short, schemes on meshes condensing in boundary layers *a priori* (see, e.g., [1–5] for partial differential equations and [6] for ordinary differential equations). The methods based on piecewise-uniform meshes condensing in boundary layers received fairly widespread use due to their simplicity and convenience in application (see, e.g., [2–5] and the references therein). The disadvantageous property

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of these numerical methods on *a priori* adapted meshes is the necessity to solve the difference equations on meshes whose step-size changes sharply in a neighbourhood of the boundary layer.

Another alternative approach to the construction of numerical methods for singularly perturbed boundary value problems developed, for example, in [7–10] leads to methods on sequentially *a posteriori* adapted meshes that are fitted (refined) in the computational process depending on the computed solution, or briefly, methods on *a posteriori adapted* meshes. In this approach, classical finite difference approximations of the boundary value problem are used; the discrete solution is corrected on a finer mesh in that subdomain where errors in the solution turn to be intolerably large. The subdomain in which the solution should be locally improved is determined using an indicator that is a functional of the solution (for example, the solution gradient) of the discrete problem. In these methods, the discrete problems on the subdomains where the solution is *a posteriori* improved are solved on uniform meshes.

In this respect, it would be of interest to consider such numerical methods on *a priori* adapted meshes in which the discrete problems in the subdomains where the computed solution is *a priori* corrected are solved on uniform meshes. Methods of this kind are unknown in the literature.

In the present paper, we consider the Dirichlet problem for a parabolic convection-diffusion equation with a small parameter ε multiplying the highest-order derivative. For the boundary value problem, we construct finite difference schemes on *locally uniform* meshes (namely, uniform meshes on the subdomains where the solution should be improved) that are adapted *a priori*, and study their convergence. To construct the schemes, a standard finite difference approximation of the differential equation is used. Note that the scheme on *a priori* condensing (in the layer) *piecewise-uniform meshes* converges ε-uniformly. The "standard" scheme on uniform meshes converges only under the condition $N^{-1} \ll \varepsilon$, where the value N defines the number of mesh points in x.

For the scheme on *a priori* adapted meshes, boundaries of the subdomains where it requires to improve the solution are determined by a *majorant for the singular component of the discrete solution*, which is specified in its turn by the perturbation parameter ε, the step-size of a mesh used in x, and also by the required accuracy of the discrete solution. On the meshes adapted with respect to the majorant function of the discrete solution, we construct a sufficiently simple finite difference scheme for which the error in the solution is weakly depending on the parameter ε. The scheme constructed on *a priori* adapted meshes converges *"almost ε-uniformly"*, precisely, under the condition $N^{-1} \ll \varepsilon^{\nu}$, where the value ν defining the scheme (the number of refinement iterations required for the discrete solution to be improved) can be chosen arbitrary in $(0,1]$.

2. Problem formulation and aim of research

2.1. Problem formulation

On the set $\overline{G}$

$$\overline{G} = G \cup S, \quad G = D \times (0,T], \tag{2.1}$$