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Finite-Difference Methods for a Class of Strongly Nonlinear Singular Perturbation Problems

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Abstract. The paper is concerned with strongly nonlinear singularly perturbed boundary value problems in one dimension. The problems are solved numerically by finite-difference schemes on special meshes which are dense in the boundary layers. The Bakhvalov mesh and a special piecewise equidistant mesh are analyzed. For the central scheme, error estimates are derived in a discrete L^1 norm. They are of second order and decrease together with the perturbation parameter ε. The fourth-order Numerov scheme and the Shishkin mesh are also tested numerically. Numerical results show ε-uniform pointwise convergence on the Bakhvalov and Shishkin meshes.

AMS subject classifications: 65L10, 65L12

Key words: Boundary-value problem, singular perturbation, finite differences, Bakhvalov and piecewise equidistant meshes, L^1 stability.

Dedicated to Professor Yucheng Su on the Occasion of His 80th Birthday

1. Introduction

We consider the following singularly perturbed boundary value problem:

$$-\varepsilon^2(k(u)u')' + c(x,u) = 0, \quad x \in I := [0,1], \quad u(0) = \alpha, \quad u(1) = \beta, \tag{1.1}$$

where ε is a small positive parameter, α and β are given constants, and the functions k and c are sufficiently smooth and satisfy

$$k^* \geq k(u) \geq k_* > 0, \quad c_u(x,u) \geq c_* > 0, \quad x \in I, \quad u \in \mathbb{R}. \tag{1.2}$$

This problem has a unique solution, u_ε, for which the following estimates hold true:

$$|u_\varepsilon^{(j)}(x)| \leq M\left(1 + \varepsilon^{-j}e^{-\gamma x/\varepsilon} + \varepsilon^{-j}e^{\gamma(x-1)/\varepsilon}\right), \quad x \in I, \quad j = 0,1,2,3,4, \tag{1.3}$$

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with a constant γ in the interval $(0, \sqrt{c_*/k^*})$. Here and throughout the paper, M is a generic positive constant independent of ε. Thus, estimates (1.3) show that the solution has in general two boundary layers whose width is $\mathcal{O}\left(\varepsilon \ln \frac{1}{\varepsilon}\right)$. This result can be proved as follows. For $K(u) = \int^u k(s)\, ds$, it holds that $K_u(u) \geq k_* > 0$, so the inverse function K^{-1} exists. We can therefore introduce the substitution $v = K(u)$ to transform (1.1) to

$$-\varepsilon^2 v'' + g(x,v) = 0, \quad x \in I, \quad v(0) = K(\alpha), \quad v(1) = K(\beta), \tag{1.4}$$

where $g(x,v) = c(x, K^{-1}(v))$. Then from $g_v(x,v) = c_u(x, K^{-1}(v))/k(u)$, we get that $g_v(x,v) > \gamma^2$. This implies that problem (1.4) has a unique solution, v_ε, and it is well known that its derivatives can be estimated by the right-hand side of (1.3). These estimates immediately transfer to u_ε.

Problems similar to (1.1), as well as the more general ones with $k = k(x,u)$, arise in applications to chemistry as models of catalytic reactions accompanied by a change in volume [3, 14, 17, 19]. Some numerical methods for those problems have been considered in [14, 17], but no complete error-analysis has been given. This is finally done in the present paper. The special case $k(u) \equiv 1$ describes the standard reaction-diffusion problem which has been discussed very often. Earlier papers, like [2, 13], typically consider the condition $c_u(x,u) \geq c_* > 0$, which is also assumed here. This condition is relaxed in [7, 8, 12, 15]. Of other more recent papers on numerical methods for singularly perturbed semilinear reaction-diffusion problems, let us mention [5] and [6]. These papers deal with *a posteriori* error estimates in the maximum norm; paper [6] is a 2D generalization of [5].

The numerical method proposed by Wang [18] for (1.1) in the non-perturbed case $\varepsilon = 1$ is the fourth-order Numerov scheme applied to (1.4). Wang considers the situation when K^{-1} can be found explicitly. Since this is not always easy to do, we discretize here the original problem after rewriting the differential equation in (1.1) as

$$-\varepsilon^2 K(u)'' + c(x,u) = 0. \tag{1.5}$$

The method we discuss in detail is the central finite-difference scheme applied on meshes of Bakhvalov and piecewise equidistant types. It is well known in the semilinear case $k(u) \equiv 1$ that the central scheme is ε-uniformly stable in the maximum norm. Here, because of the strong nonlinearity of the problem, it is much easier to use a discrete L^1 norm to prove stability uniform in ε. Stability of finite-difference approximations of quasilinear singular perturbation problems is often proved in this norm, see [1] for instance. Solutions of such problems may have interior layers with *a priori* unknown locations. This is not the case in the present problem, but, in addition to the strong nonlinearity, there is another reason for using the L^1 norm. If $w(x) = \exp(-\gamma x/\varepsilon)$ is the exponential boundary-layer function, then $\|w\|_1$ is of order ε, thus small values of ε increase accuracy in L^1 norm. Such higher L^1-accuracy is important in the catalytic-reaction applications when calculating the so-called efficiency factor, see [17].

ε-uniform stability in L^1 norm implies convergence results in the same norm, the errors being estimated by

$$E_B := MN^{-2}\left(\varepsilon + e^{-mN}\right) \quad \text{on the Bakhvalov mesh} \tag{1.6}$$