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Cubic Spiral Transition Matching G^2 Hermite End Conditions

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Abstract. This paper explores the possibilities of very simple analysis on derivation of spiral regions for a single segment of cubic function matching positional, tangential, and curvature end conditions. Spirals are curves of monotone curvature with constant sign and have the potential advantage that the minimum and maximum curvature exists at their end points. Therefore, spirals are free from singularities, inflection points, and local curvature extrema. These properties make the study of spiral segments an interesting problem both in practical and aesthetic applications, like highway or railway designing or the path planning of non-holonomic mobile robots. Our main contribution is to simplify the procedure of existence methods while keeping it stable and providing flexile constraints for easy applications of spiral segments.

AMS subject classifications: 65D05, 65D07, 65D10, 65D17, 65D18

Key words: Path planning, spiral, cubic Bézier, G^2 Hermite, Computer-Aided Design (CAD), computational geometry.

1. Introduction

The research and development in the area of path planning include theoretical and practical aspects of geometric modeling approaches. Fair path planning is one of the fundamental problems, with numerous applications in the fields of science, engineering and technology such as highway or railway route designing, networks, robotics, GIS, navigation, CAD systems, collision detection and avoidance, animation, environmental design, communications, image processing, digital data compression, gear designing, and other disciplines [4, 15]. One of the main approaches to path planning is through the use of spline and spiral functions.

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In path planning it is often required to have a planar transition curve of J-shaped (between straight line and circle), U-shaped (between two straight lines), C-shaped (between two circles), and an S-shaped (between two circles) [6,7,9,18]. Planer spirals have monotonic curvature with constant sign, therefore such segments are free from singularities, inflection points, and curvature extrema. It is considered desirable to have such a curve in satellite path planning, highway or railway route designing, GIS, or non-holonomic (car-like) robot path planning [13,14]. In curve and surface design, it is mostly required to have a spiral curve of G^2 contact matching Hermite conditions, i.e., under the fixed positional, tangential, and curvature end conditions. These properties make spiral curves important for both physical [4] and aesthetic applications [1].

Dietz and Piper [2] have numerically computed values to aid in adjusting the selection of control points with the help of tables for building cubic spiral curve matching G^2 Hermite conditions. However, their numerically obtained spiral regions would become smaller or even empty as the tangent angles at end points become relatively equal. An extension of this method is discussed in [3] by using rational cubic function. Their method gives more spiral regions but the case of relatively equal tangent angles is not discussed. Further, another case of a very small tangent angle at start point leads to a spike near end point in their scheme.

Recently, Habib and Sakai [10] used rational cubic spline to find reachable regions for the spiral segment matching G^2 Hermite conditions. In their method, spiral conditions are derived on the whole segment by using derivative of curvature of polynomial. A free parameter is used to find the admissible region for a spiral segment with respect to the curvatures at its endpoints under the fixed positional and tangential end conditions. Although their method is stable and preserving all the geometric features to overcome the problems in [2, 3] but the analysis on finding the spiral regions is computationally very expensive (of degree 12 of derivative of curvature) due to the rational form of cubic polynomial.

This paper extends and simplifies the analysis of [2, 3, 8, 10, 16, 17] by using a single segment of cubic Bézier function for a C-shaped spiral segment matching G^2 Hermite end conditions. Our main contribution is the achievement of almost the same spiral regions by using a cubic polynomial in non-rational form. Computational cost of finding spiral regions is also very low because the degree of derivative of its curvature is just five (instead of twelve). Since spiral conditions are derived on the whole segment, our proposed method is stable and there is no fear of spiking phenomenon of non-monotone curvature.

2. Background

2.1. Notations and conventions

The usual Cartesian co-ordinate system is presumed. Boldface is used for points and vectors, e.g.,

$$\boldsymbol{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}. \tag{2.1}$$