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A FRONT-FIXING FINITE ELEMENT METHOD FOR THE VALUATION OF AMERICAN PUT OPTIONS ON ZERO-COUPON BONDS

ANTHONY D. HOLMES AND HONGTAO YANG

Abstract. A front-fixing finite element method is developed for the valuation of American put options on zero-coupon bonds under a class of one-factor models of short interest rates. Numerical results are presented to examine our method and to compare it with the usual finite element method. A conjecture concerning the behavior of the early exercise boundary near the option expiration date is proposed according to the numerical results.

Key Words. American put option, zero-coupon bond, free boundary problem, front-fixing method, finite element method

1. Introduction

Consider a class of one-factor models of the short interest rate process:

$$r(t) = \zeta(X(t)), \quad dX(t) = (\phi(t) - \psi(t)X(t))dt + \sigma(t)dW(t), \tag{1.1}$$

where $\zeta(x)$ is an invertible function on $(-\infty, +\infty)$, $\phi(t)$, $\psi(t)$ and $\sigma(t)$ are some known functions of t, and $W(t)$ is a standard Brownian motion under the risk-neutral measure. For $\zeta(x) = x$ and $\zeta(x) = e^x$, we have the popular Hull-White model ([7]) and Black and Karasinski model ([4]), respectively.

Let $x = \eta(r)$ be the inverse function of $r = \zeta(x)$. Assume that $\zeta(x)$ is twice continuously differentiable. By using Ito's formula, we can obtain the stochastic differential equation (SDE) for the interest rate process $r(t)$:

$$dr(t) = a(r(t), t)dt + b(r(t), t)dW(t), \tag{1.2}$$

where

$$a(r, t) = \zeta'(\eta(r))(\phi(t) - \psi(t)\eta(r)) + \frac{1}{2}\sigma(t)^2\zeta''(\eta(r)), \quad b(r, t) = \sigma(t)\zeta'(\eta(r)).$$

Then we have the following fundamental partial differential equation (PDE) for the rational price $V(r, t)$ of an interest rate derivative at time t ([3][14]):

$$V_t + \frac{1}{2}b(r, t)^2V_{rr} + a(r, t)V_r - rV = 0. \tag{1.3}$$

Since $\zeta(x)$ is invertible, we can rewrite the above PDE into the PDE for $\widetilde{V}(x, t) = V(\zeta(x), t)$:

$$\widetilde{V}_t + \frac{1}{2}\sigma(t)^2\widetilde{V}_{xx} + (\phi(t) - \psi(t)x)\widetilde{V}_x - \zeta(x)\widetilde{V} = 0.$$

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The assumption that $\zeta(x)$ is invertible is necessary to derive the equivalent SDE (1.2) for $r(t)$ and the PDE (1.3) . For example, when $\zeta(x) = x^2$, we have the well-known quadratic model. Since $\zeta(x) = x^2$ is not invertible on $(-\infty, +\infty)$, we do not have an SDE for the interest rate process $r(t) = X(t)^2$ and can not express the interest rate derivative price as a function of interest rate r. It should be pointed out that $\zeta(x)$ can be chosen to be any bounded invertible function from $(-\infty, +\infty)$ to $(0, 1)$, e.g.,

$$\zeta(x) = \frac{e^x}{1 + e^x}. \tag{1.4}$$

For such a choice of $\zeta(x)$, the interest rates will not take unrealistic values more than 1. We are referred to [8] and [9] for other possible choices of $\zeta(x)$ and the calibration of one-factor models.

Now let us consider an American put option on a T^*-maturity zero-coupon bond. The option expiration date is T $(< T^*)$ and its exercise price is K. Since the option can be exercised at any time up to its expiration date, there is a critical interest rate $r^*(t)$ which is referred to as the early exercise interest rate. Denote the option price by $p(r, t)$. Let $x^*(t) = \eta(r^*(t))$ and $\widetilde{p}(x, t) = p(\zeta(x), t)$. According to the above argument, we can show that $\widetilde{p}(x, t)$ and $x^*(t)$ solve the following free boundary problem:

$$\widetilde{p}_t + \frac{1}{2}\sigma(t)^2 \widetilde{p}_{xx} + (\phi(t) - \psi(t)x)\widetilde{p}_x - \zeta(x)\widetilde{p} = 0, \ -\infty < x < x^*(t), \ 0 \le t \le T, \tag{1.5}$$

$$\widetilde{p}(x^*(t), t) = g(x^*(t), t), \quad 0 \le t \le T, \tag{1.6}$$

$$\widetilde{p}_x(x^*(t), t) = g_x(x^*(t), t), \quad 0 \le t \le T, \tag{1.7}$$

$$\widetilde{p}(x, T) = g(x, T), \quad -\infty < x < \infty, \tag{1.8}$$

where $g(x, t) = \max(K - \widetilde{P}(x, t; T^*), 0)$ is the payoff of the put option and $\widetilde{P}(x, t; T^*)$ is the bond price when $r = \zeta(x)$ at time t.

Front-fixing/front-tracking methods have been applied for numerical valuation of American options. Their favorable feature is that the early exercise boundaries and option prices can be computed simultaneously and with higher accuracy. We are referred to [6, 11, 12, 13, 15, 16] for recent work in this aspect for American stock options. For the usual front-fixing method, the Landau transformation $y = (x + L)/(x^*(t) + L)$ will be employed after restricting the problem on the bounded domain $(-L, x^*(t))$ for a sufficiently large positive number L. Here we shall use the linear transformation $y = x + L - x^*(t)$ while the problem is truncated on the variable domain $(x^*(t) - L, x^*(t))$. The transformation will not affect the coefficient of the leading term in the partial differential equation (1.5). This approach is first proposed for American options on stocks in [2], and our numerical results show that it produces much more accurate approximations of early exercise boundaries and option prices. In this paper we shall consider such a front-fixing finite element method (FFEM) for the free boundary problem (1.5)–(1.8).

The outline of the paper is as follows. In §2 we develop a FFEM for the free boundary problem (1.5)–(1.8) and establish its stability with an appropriate assumption. In §3 we give details for the implementation of our method and show how to compute bond prices and their derivatives when analytic formulas are not available. In §4 numerical results are presented to examine our method and to compare it with the usual finite element method in [17]. In particular, we shall analyze the behavior of early exercise interest rates near the option expiration dates numerically. We conclude the paper with remarks in the last section, §5.