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A MULTILEVEL METHOD FOR SOLVING THE HELMHOLTZ EQUATION : THE ANALYSIS OF THE ONE-DIMENSIONAL CASE

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Abstract. In this paper we apply and discuss a multilevel method to solve a scattering problem. The multilevel method belongs to the class of incremental unknowns method as in [10]; in this work, the best performance was obtained with a coarsest grid having roughly two points per linear wavelength. We analyze this method for a simple model problem following H. Yserentant [17]. In this case, the main limitation to multilevel methods is closely linked to the indefiniteness of the Helmholtz problem.

Key words. hierarchical basis, indefinite problem, Helmholtz equation, iterative methods

1. Introduction

In this paper we are interested in applying the strategy introduced by H. Yserentant [17] to solve an indefinite elliptic boundary value problem that comes from acoustics [7], [6]. This problem, leading to a non-coercive bilinear form, reads as follows

$$-u_{xx} - k^2 u = f, \quad \text{in }]0,1[\tag{1.1}$$

$$u(0) = 0, \tag{1.2}$$

$$u_x(1) = \imath k u(1). \tag{1.3}$$

Here we adopt the notations $\imath = \sqrt{-1}$, while the wavenumber $k = \frac{\omega L}{c}$ is a positive real number (supposed larger than 1 in the sequel). Here some scaling has been performed; this problem occurs when one considers a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ that moves from the left to the right boundary, whose frequency is ω and that satisfies some Sommerfeld radiation condition at $+\infty$. For numerics, one tracks this solution on a box $[0, L]$, and after scaling in space, this condition (1.3) replaces $u(x) \sim e^{\imath k x}$ at $+\infty$.

This one-dimensional problem belongs to exterior boundary value problems of the form

$$-\Delta u - k^2 u = f \quad \text{in } \Omega \tag{1.4}$$

$$u = g \quad \text{on } \Gamma \subset \partial\Omega \tag{1.5}$$

$$\mathcal{F}u = 0 \quad \text{on } \partial\Omega \tag{1.6}$$

where the operator $\mathcal{F}$ corresponds to the chosen absorbing boundary condition (ABC), while the second equation depends of the (acoustic) properties of the scatterer. The Helmholtz problem at hand is expected to produce a solution with an oscillatory behavior on the wavelength ($\lambda = 2\pi/k$) scale. The analysis conducted hereafter should be extended to the two-dimensional or three-dimensional problem with an approximation of first order ABC without any other difficulties than technical ones. Otherwise, if one considers the problem with Sommerfeld radiation

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Computational tests have been performed using Orca, the server of the Centre Commun de Calcul Intensif of the Université des Antilles et de la Guyane.

condition at infinity, one must use a different framework as weighted Sobolev spaces, but working without the assumptions needed to validate the Poincaré inequality might be hard.

Multilevel methods, like hierarchical basis [15, 16] for finite element approximation or incremental unknowns [12] in finite difference context, are effective for the numerical solution of many partial differential equations. They seem being almost so robust and powerful as classical multigrid methods for solving elliptic partial differential equations [3]. Nevertheless, for large scales problems, multilevel approaches do not apply straightforwardly and more involved method have to be considered [9, 1, 8]. For some classes of problems like ours, it has been proven that the effectiveness of classical multigrid methods often fails [13, 4, 5]. In particular, the indefiniteness of the discrete problems is certainly the main reason for which the coarsest grid must be not too coarse.

The multigrid methods by combining interpolation and pre and post-smoothing catch step by step the harmonics of the solution whereas the multilevel methods involve the projection of the solution onto a Krylov space in the multilevel basis. Even if the approach of the latter looks quite far away from the classical multigrid methods, we will prove that they have a similar limitation onto the sparsity of the coarsest level of grid for indefinite discrete problems (this result has been pointed out in [10]).

For the Helmholtz problem under consideration, despite the fact that the associated bilinear form is not positive definite, one can exhibit a large subspace W of the energy space with a finite co-dimension (which varies as k^4), and such that the bilinear form restricted to W becomes coercive. Hence, dealing with finite element multilevel approximation of the equation, we develop the strategy introduced by H. Yserentant to trap the bad behavior of the bilinear form on a finite element space corresponding to a coarse grid approximation of the equation, and then to proceed to multilevel analysis on finer grids. The significant drawback of our method is that it does not apply to very high frequency problems since the magnitude of the coarse grid behaves as k^4.

The outline of this paper is as follows. In section 2, we introduce the one-dimensional model problem and study its properties. The section 3 is devoted to its approximation by multilevel finite element (which is similar to the incremental unknowns in finite differences). One shows in particular the influence of the indefiniteness of the problem onto the discrete problem. Computations of the condition number of the stiffness matrix for the hierarchical basis are given in section 4, in agreement with the analysis.

Let us complete this introduction by some notations. To treat the absorbing boundary condition (1.3), we need to use complex valued functions. Furthermore, to adapt the guidelines introduced in [17] to complex valued functions, we consider $L^2(0,1)$ the real-Hilbert space whose scalar product is

$$(u,v) = \mathrm{Re}\int_0^1 u(x)\overline{v(x)}dx = \mathrm{Re}\int u\bar{v}dx. \tag{1.7}$$

Note that one omits to write generic constant that may vary from one line to another one, but that is independent of k and of h_0, h, denote respectively the mesh size of the coarse and fine grid approximation of the problem. We also use the space $V = \{u \in L^2(0,1); u_x \in L^2(0,1) \text{ and } u(0) = 0\}$. For the sake of conciseness, we