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A Matrix-Vector Operation-Based Numerical Solution Method for Linear m-th Order Ordinary Differential Equations: Application to Engineering Problems

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Abstract. Many problems in engineering sciences can be described by linear, inhomogeneous, m-th order ordinary differential equations (ODEs) with variable coefficients. For this wide class of problems, we here present a new, simple, flexible, and robust solution method, based on piecewise exact integration of local approximation polynomials as well as on averaging local integrals. The method is designed for modern mathematical software providing efficient environments for numerical matrix-vector operation-based calculus. Based on cubic approximation polynomials, the presented method can be expected to perform (i) similar to the Runge-Kutta method, when applied to stiff initial value problems, and (ii) significantly better than the finite difference method, when applied to boundary value problems. Therefore, we use the presented method for the analysis of engineering problems including the oscillation of a modulated torsional spring pendulum, steady-state heat transfer through a cooling web, and the structural analysis of a slender tower based on second-order beam theory. Related convergence studies provide insight into the satisfying characteristics of the proposed solution scheme.

AMS subject classifications: 34A30, 34B60

Key words: Numerical integration, polynomial approximation, ODE, variable coefficients, initial conditions, boundary conditions, stiff equation.

1 Introduction

Many problems in engineering sciences can be described by linear, inhomogeneous, m-th order, ordinary differential equations (ODEs) with variable coefficients

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$$\sum_{i=0}^{m} a_i(x) y^{(i)}(x) = r(x), \qquad y^{(i)}(x) \stackrel{\text{def}}{=} \frac{d^i y(x)}{dx^i}. \tag{1.1}$$

This includes problems such as oscillations, one-dimensional heat transfer, or beam bending. Classical numerical solution methods for related initial value problems and boundary value problems, respectively, include the backward Euler method, the Runge-Kutta method, Adams-Bashforth-Moulton methods [1–3], backward differentiation formulas (BDFs) [4], Rosenbrock methods [5,6], as well as the finite difference method.

During the last decades, quite a number of new methods have been presented for specific classes of problems. For stiff first-order initial value problems, numerical differentiation formulas (NDFs) and a modified Rosenbrock method were presented in [7]. Meshless local Petrov-Galerkin methods [8, 9] have become popular for solving thin beam problems in the framework of the first-order theory. Discontinuous Galerkin methods [10] were shown to be efficient for second-order ODEs with periodic coefficient functions and periodic perturbation functions. Piecewise linearized methods [11] have turned out to be useful for non-stiff initial value problems. The differential quadrature method [12, 13] has gained interest of researchers dealing with dynamic problems, including three-dimensional vibration of functionally graded circular plates analyzed based on the one-dimensional differential quadrature method [14]. Also integration schemes based on approximations provided by radial basis functions have become popular [15,16].

When selecting a specific method out of the variety of available approaches capable to solve problems subsumable under (1.1), there is typically a trade-off between numerical efficiency and versatility. There exist highly efficient methods which are designed for specific classes of problems, i.e., for ODEs with a specific order of differentiation m, for specific types of coefficient functions $a_i(x)$, for specific perturbation functions $r(x)$, and for specific types of boundary conditions; but such specialized methods frequently exhibit a limited utilizability for other specific problems. Methods designed for initial value problems, for instance, may lack user friendliness when applying them to general boundary value problems which include boundary conditions referring to the end of the integration interval. Highly robust and flexible methods, in turn, which are generally applicable to virtually all problems subsumable under (1.1), commonly cannot compete with the aforementioned, highly efficient methods, when a specific problem at hand has to be solved.

This is the motivation to present a new, simple, flexible, and robust method for the solution of ordinary differential equations such as (1.1), based on piecewise exact integration of local approximation polynomials as well as on averaging local integrals. The method is designed to comply with the following requirement profile:

- The method should be applicable to any linear, inhomogeneous, m-th order ODE with variable coefficients – either associated with an initial value problem, or with a