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THE α METHOD FOR SOLVING DIFFERENTIAL ALGEBRAIC INEQUALITY (DAI) SYSTEMS

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Abstract. This paper describes an algorithm for "direct numerical integration" of the initial value Differential-Algebraic Inequalities (DAI) in a time stepping fashion using a sequential quadratic programming (SQP) method solver for detecting and satisfying active path constraints at each time step. The activation of a path constraint generally increases the condition number of the active discretized differential algebraic equation's (DAE) Jacobian and this difficulty is addressed by a regularization property of the α method. The algorithm is locally stable when index 1 and index 2 active path constraints and bounds are active. Subject to available regularization it is seen to be stable for active index 3 active path constraints in the numerical examples. For the high index active path constraints, the algorithm uses a user-selectable parameter to perturb the smaller singular values of the Jacobian with a view to reducing the condition number so that the simulation can proceed. The algorithm can be used as a relatively cheaper estimation tool for trajectory and control planning and in the context of model predictive control solutions. It can also be used to generate initial guess values of optimization variables used as input to inequality path constrained dynamic optimization problems. The method is illustrated with examples from space vehicle trajectory and robot path planning.

Key Words. Differential-algebraic equations, Trajectory planning, Numerical optimization, Inequality path constraints

1. Background

Many engineering control problems, especially those with inequality path constraints yield Differential-Algebraic Inequalities (DAI) models. The need for solving a DAI system arises in robotic path planning [19], safety envelope [10] and trajectory [7] generation, in model predictive control approaches [8], and in voltage control of electrical equipments [11]. Traditionally, a DAI is handled, almost always, in the context of optimal control problems where the inequality path constraints in the discretized optimal control problem are handled by the optimizer as inequality constraints at each mesh point (e.g., in Direct Transcription and collocation schemes) or as a sub-interval wise cumulative error integral that is minimized (e.g., in multiple shooting schemes).

However, in certain situations the direct simulation of a DAI does arise. For example, when interior point methods are used for dynamic optimization problems,

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the initial guess of a feasible solution involving the satisfaction of the DAI system is needed. Intermediate level trajectory planning through constraint programming also entails the necessity of a DAI solver [19].

Available Differential-Algebraic Equation (DAE) solvers are limited to addressing inequalities only in the form of positivity constraints on the states and controls (e.g., DASPK [3]). Some DAE solvers can detect whether a constraint has become active with root-finding techniques (e.g., DASRT [3], DASPKE [13]) and mainly concern with DAE systems with discontinuities.

Examples of DAI integrators are few. A solver with constraint smoothing and local planning has been described in [19]. This algorithm checks for weakly approximate safety condition of using the control values from the previous time step and proceeds with activating a buffer zone as a path inequality constraint becomes nearly active. A barrier function minimization is used if the safety was violated to obtain a new set of initial guesses for the controls at that time step. Then the dynamics is integrated provided the states obtained satisfy the bounds and inequality path constraints.

1.1. Introduction to the present work. At every time step a DAI integrator must

- detect active path constraints
- determine which algebraic variables control on to the active path constraints
- handle the possible numerical row rank deficiency in the active constraints Jacobian in the SQP method. The numerical rank deficiency may occur due to activation of high differential index (see definition in [3] and called the index hereafter) active path constraints and due to abrupt changes in the active path constraint set.

The present algorithm addresses the above requirements as follows.

- A standard sequential quadratic programming (SQP) method that is used as a solver at each time step detects the active path constraints.
- The SQP method also determines which algebraic variables control on to the active path constraints by constructing a square basis matrix (defined in section 3.1) which has the least condition number over available column permutations in the active constraints Jacobian in the SQP method.
- The numerical row rank deficiency is addressed by varying a parameter in the DAI discretization which leads to increase in the smallest singular values (i.e., regularization) of the basis matrix.

The DAI solver finds a feasible solution locally in contrast to the Multiple Shooting method or the Direct Transcription method where the dynamic optimization problem discretized over the entire simulation interval enters the QP iterations of an SQP solver. The trade-off for a DAI solver is cheaper computational cost in finding a feasible solution one time step at a time involving much smaller matrices than the dynamic optimization.

The method is intended to be either a cheap tool for generating feasible initial guess for the dynamic optimization problem solved by Multiple Shooting or Direct Transcription with an interior point method, or to be a solver for rapid trajectory planning via constraint programming at an intermediate specification level. It is not meant to replace methods that find solutions over the entire simulation interval, such as the Multiple Shooting or Direct Transcription methods.

The regularization property of the present algorithm is different from the existing approaches of path constraint perturbation and/or modification (such as the