INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 7, Number 4, Pages 656–666

LOCAL PROJECTION FINITE ELEMENT STABILIZATION FOR DARCY FLOW

KAMEL NAFA

Abstract. Local projection based stabilized finite element methods for the solution of Darcy flow offer several advantages as compared to mixed Galerkin methods. In particular, the avoidance of stability conditions between finite element spaces, the efficiency in solving the reduced linear algebraic system, and the convenience of using equal order continuous approximations for all variables. In this paper we analyze the pressure gradient method for Darcy flow and investigate its stability and convergence properties.

Key Words. Stabilized finite elements, Darcy equations, convergence, error estimates.

1. Introduction

Numerical methods for Darcy equations are traditionally-based on a primal single field formulation for the pressure or on the mixed two field velocity-pressure formulation. It is well known that the choice of the finite element spaces, for the mixed formulation, is subject to the inf-sup stability condition ([10]). This has lead to the use of classical mixed Raviart-Thomas and Brezzi-Douglas-Marini finite elements ([10]). This approach though giving good accuracy for both velocity and pressure ([20]) has its draw back complexity.

It has been a few years since stabilized finite element methods have been extended to the Darcy equations (see, [23], [5], [6], and [12]). Despite the fact that such methods are well established for fluid flow problems based on Stokes-like operator (see, [19], [17], [32], [7], [3], [16], [21], and [22]). In [23] a term based on the residual of Darcy law is added to the classical Galerkin formulation making the formulation stable for all combination of conforming continuous velocity-pressure approximations. Another class of stabilized methods has been derived using Galerkin methods enriched with bubble functions (see, [1] and [2]). Alternative stabilization techniques based on a least squares formulation have been proposed by ([5]), and ([6]).

Recently, local projection methods that seem less sensitive to the choice of parameters and have better local conservation properties were proposed for Stokes problem (see, [14], and [4]). The two-level pressure gradient method with a projection onto a discontinuous finite element space of a lower degree defined on a coarser grid has been analyzed in [4], [8], [25], [26], and [12]. We note that although the two-level pressure gradient stabilization method gives a slightly bigger discretisation stencil, the drawback is not severe because the pressure-gradient unknowns can be eliminated locally.

Received by the editors February 9, 2009 and, in revised form, June 10, 2009.
2000 *Mathematics Subject Classification.* 65N12, 65N30, 65N15, 76D07.
This research was supported by Sultan Qaboos University, Project IG/SCI/DOMS/09/12.

In this paper we analyze the pressure gradient stabilization method for the Darcy equations. As in [29], [30], [27] and [28], the stability of the pressure-gradient method is proved by constructing an interpolant with additional orthogonality property with respect to the projection space. As a result, optimal rates of convergence are found for the velocity and pressure approximations.

2. Variational formulation

Let Ω be a bounded open region of $\mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$. Darcy's law for the flow of a viscous fluid in a permeable medium, and conservation of mass are written as follows

$$\mathbf{u}+\nabla p = \mathbf{0} \quad \text{in } \Omega \tag{1}$$
$$\nabla\cdot\mathbf{u} = f \quad \text{in } \Omega \tag{2}$$
$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega \tag{3}$$

where, $\mathbf{u}$ is the Darcy velocity vector, p is the pressure, and $\mathbf{n}$ the outward normal vector.

Let

$$\mathbf{V} = \mathbf{H}_0(div,\Omega) = \left\{\mathbf{v}\in\left[L^2(\Omega)\right]^2 : \nabla\cdot\mathbf{v}\in L^2(\Omega),\ \mathbf{u}\cdot\mathbf{n}=0 \text{ on } \partial\Omega\right\}$$
$$Q = H^1(\Omega)\cap L^2_0(\Omega)$$

where $L^2_0(\Omega)$ denotes the set of square integrable functions with null average.

Define the forms

$$A(\mathbf{u},p;\mathbf{v},q) = (\mathbf{u},\mathbf{v})-(p,\nabla\cdot\mathbf{v})+(q,\nabla\cdot\mathbf{u}) \tag{4}$$

and

$$F(\mathbf{v},q) = (f,q), \tag{5}$$

for all $(\mathbf{v},q)\in\mathbf{V}\times Q$, with $(\cdot,\cdot)$, as usual, denoting the L^2-inner product on the region Ω.

Then, the weak formulation of (1)-(3) reads in compact notation as

$$A(\mathbf{u},p;\mathbf{v},q) = F(\mathbf{v},q)\ ,\quad \forall(\mathbf{v},q)\in\mathbf{V}\times Q. \tag{6}$$

A natural norm for the above problem is

$$\|(\mathbf{u},p)\|_D = \|\mathbf{u}\|^2_{0,\Omega}+\|\nabla.\mathbf{u}\|^2_{0,\Omega}+\|p\|^2_{0,\Omega}\,. \tag{7}$$

Let $\mathbf{V}_h$ and Q_h be finite dimensional subspaces of $\mathbf{V}$ and Q, respectively. Then, the classical Galerkin discrete problem reads

Find $(\mathbf{u}_h,p_h)\in\mathbf{V}_h\times Q_h$ such that:

$$A(\mathbf{u}_h,p_h;\mathbf{v},q) = F(\mathbf{v},q)\ ,\quad \forall(\mathbf{v},q)\in\mathbf{V}_h\times Q_h. \tag{8}$$

Note that formulation (8) is stable and accurate only for velocity and pressure approximations satisfying the inf-sup condition (see, for example [10]). In particular, this condition rules out low equal-order C^0 approximations of the pressure and velocity.