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ON THE OPTIMAL CONTROL PROBLEM OF LASER SURFACE HARDENING

NUPUR GUPTA, NEELA NATARAJ AND AMIYA K. PANI

Abstract. We discuss an optimal control problem of laser surface hardening of steel which is governed by a dynamical system consisting of a semilinear parabolic equation and an ordinary differential equation with a non differentiable right hand side function f_+. To avoid the numerical and analytic difficulties posed by f_+, it is regularized using a monotone Heaviside function and the regularized problem has been studied in literature. In this article, we establish the convergence of solution of the regularized problem to that of the original problem. The estimates, in terms of the regularized parameter, justify the existence of solution of the original problem. Finally, a numerical experiment is presented to illustrate the effect of regularization parameter on the state and control errors.

Key Words. Laser surface hardening of steel, semilinear parabolic equation, ODE with non-differentiable forcing function, regularized Heaviside function, regularised problem, convergence with respect to regularization parameter, numerical experiments.

1. Introduction

In this paper, we discuss an optimal control problem described by the laser surface hardening of steel. The purpose of surface hardening is to increase the hardness of the boundary layer of a workpiece by rapid heating and subsequent quenching (see Figure 1). The desired hardening effect is achieved as the heat treatment leads to a change in micro structure. A few applications include cutting tools, wheels, driving axles, gears, etc. Let $\Omega \subset \mathbb{R}^2$, denoting the workpiece, be a convex, bounded domain with piecewise Lipschitz continuous boundary $\partial\Omega$, $Q = \Omega \times I$ and $\Sigma = \partial\Omega \times I$, where $I = (0, T)$, $T < \infty$. Following Leblond and Devaux[7], the evolution of volume fraction of the austenite $a(t)$ for a given temperature evolution $\theta(t)$ is described by the initial value problem:

$$\partial_t a = f_+(\theta, a) = \frac{1}{\tau(\theta)}[a_{eq}(\theta) - a]_+ \text{ in } Q, \tag{1}$$

$$a(0) = 0 \text{ in } \Omega, \tag{2}$$

where $a_{eq}(\theta(t))$, denoted as $a_{eq}(\theta)$ for notational convenience, is the equilibrium volume fraction of austenite and τ is a time constant. The term $[a_{eq}(\theta) - a]_+ =$

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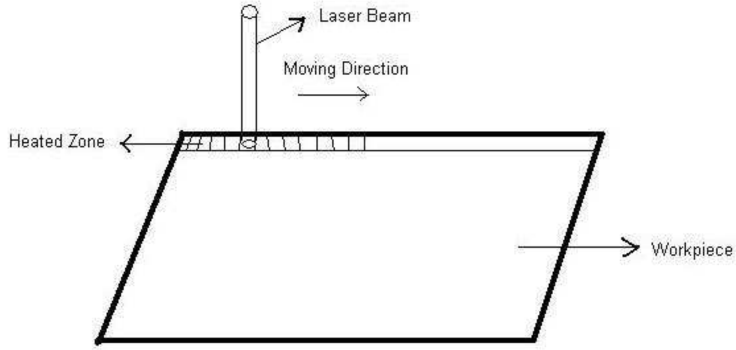

FIGURE 1. Laser Hardening Process

$(a_{eq}(\theta) - a)\mathcal{H}(a_{eq}(\theta) - a)$, where $\mathcal{H}$ is the Heaviside function

$$\mathcal{H}(s) = \begin{cases} 1 & s > 1 \\ 0 & s \leq 0, \end{cases}$$

denotes the non-negative part of $a_{eq}(\theta) - a$, that is,

$$[a_{eq}(\theta) - a]_+ = \frac{(a_{eq}(\theta) - a) + |a_{eq}(\theta) - a|}{2}.$$

Neglecting the mechanical effects and using the Fourier law of heat conduction, the temperature evolution can be obtained by solving the following energy balance equation:

$$\rho c_p \partial_t \theta - K \triangle \theta = -\rho L \partial_t a + \alpha u \quad \text{in } Q, \tag{3}$$

$$\theta(0) = \theta_0 \quad \text{in } \Omega, \tag{4}$$

$$\frac{\partial \theta}{\partial n} = 0 \quad \text{on } \Sigma, \tag{5}$$

where the density ρ, the heat capacity c_p, the thermal conductivity K and the latent heat L are assumed to be positive constants. Further, θ_0 denotes the initial temperature. The term $u(t)\alpha(x,t)$ describes the volumetric heat source due to laser radiation and the laser energy $u(t)$ is a time dependent control variable. Since the main cooling effect is a self-cooling of the workpiece, a homogeneous Neumann condition is assumed on the boundary.

To maintain the quality of the workpiece surface, it is important to avoid the melting of the surface. In the case of laser hardening, it is a quite delicate problem to obtain parameters that avoid melting but nevertheless lead to the right amount of hardening. Mathematically, this corresponds to an optimal control problem in which we minimize the cost functional defined by:

$$J(\theta, a, u) = \frac{\beta_1}{2} \int_\Omega |a(T) - a_d|^2 dx + \frac{\beta_2}{2} \int_0^T \int_\Omega [\theta - \theta_m]_+^2 dx ds + \frac{\beta_3}{2} \int_0^T |u|^2 ds \tag{6}$$

$$\text{subject to } (1) - (5) \text{ in the set of admissible controls } U_{ad}, \tag{7}$$

where $U_{ad} = \{v \in U : \|v\|_U \leq M, \quad \text{for fixed positive } M\}$ with $U = L^2(I)$, β_1, β_2 and β_3 are positive constants and a_d is the given desired fraction of the austenite. The second term in (6) is a penalizing term that penalizes the temperature below the melting temperature θ_m.

The mathematical model for the laser surface hardening of steel has been studied in [4] and [7]. For an extensive survey on mathematical models for laser material