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APPROXIMATE FORMULAE FOR PRICING ZERO-COUPON BONDS AND THEIR ASYMPTOTIC ANALYSIS

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Abstract. We analyze analytic approximation formulae for pricing zero-coupon bonds in the case when the short-term interest rate is driven by a one-factor mean-reverting process with a volatility nonlinearly depending on the interest rate itself. We derive the order of accuracy of the analytical approximation due to Choi and Wirjanto. We furthemore give an explicit formula for a higher order approximation and we test both approximations numerically for a class of one-factor interest rate models.

Key Words. One factor interest rate model, Cox-Ingersoll-Ross model, bond price, analytical approximation formula, experimental order of convergence.

1. Introduction

Term structure models give the dependence of time to maturity of a discount bond and its present price. One-factor models are often formulated in terms of a stochastic differential equation for the instantaneous interest rate (short rate). In the theory of nonarbitrage term structure models the bond prices (yielding the interest rates) are given by a solution to a parabolic partial differential equation. The stochastic differential equation for the short rate is specified either under a real (observed) probability measure or risk-neutral one. A risk-neutral measure is an equivalent measure such that the derivative prices (bond prices in particular) can be computed as expected values. If the short rate process is considered with a real probability measure, a function λ describing the so-called market price of risk has to be provided. The volatility part of the process is the same for both real and risk-neutral specification of the process. The changes in the drift term depend on the so called market price of risk function λ.

It is often assumed that the short rate evolves according to the following mean reverting stochastic differential equation

$$\mathrm{d}r = (\alpha + \beta r)\mathrm{d}t + \sigma r^{\gamma}\mathrm{d}w \tag{1}$$

where $\sigma > 0$, $\gamma \geq 0$, $\alpha > 0$, β are given parameters. In particular, it includes the well known Vasicek model ($\gamma = 0$) and Cox-Ingersoll-Ross model ($\gamma = 1/2$) (c.f. Vasicek (7) and Cox, Ingersoll and Ross (3)). For those particular choices of γ closed form solutions of the bond pricing PDE (2) are known. Assuming a suitable form of the market price of risk it turns out that both the real and risk neutral processes for the short rate have the form (1). More details concerning the term structure modeling can be found in Kwok (4).

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Using US Treasury Bills data (June 1964 - December 1989), the real probability model (1) and generalized method of moments Chan *et al.* (2) estimated the parameter γ at the value 1.499. This is considered to be an important contribution, as it drew attention to a more realistic form of the short rate volatility (compared to Vasicek or CIR models). Using the same US Treasury Bills data, Nowman in (5) estimated $\gamma = 1.361$ by means of Gaussian methodology. It should be noted that these estimations of γ are beyond values $\gamma = 0$ or $\gamma = \frac{1}{2}$ for which the closed form solution of the bond prices is known in an explicit form. In (6) a model with interest rates from eight countries using generalized method of moments and quasi maximum likelihood method has been estimated. They tested the restrictions imposed by Vasicek and CIR models using the J-statistics in the generalized method of moments and likelihood ratio statistics in the quasi maximum likelihood method. In all tested cases except of one, the restrictions $\gamma = 0$ or $\gamma = \frac{1}{2}$ were rejected. Hence, the study of the bond prices for values of γ different from 0 and 1/2 can be justified by empirical results. However, in these cases no closed form expression for bond prices is known. An approximate analytical solution was suggested in (1) which could make the models with general $\gamma > 0$ to be more widely used. In this paper, we analyze the analytical approximation by Choi and Wirjanto (1) and derive its accuracy order. Furthemore, by adding extra terms to it we derive an improved, higher order approximation of the bond prices.

The paper is organized as follows. In the second section, we derive the order of approximation of the analytical approximative solution from (1). We derive a new, higher order accurate approximation. In the third section, we compare the two approximations with a known closed form solution from the CIR model ($\gamma = \frac{1}{2}$). In Appendix we provide a proof of uniqueness of a solution of a partial differential equation for bond pricing for the parameter range $\frac{1}{2} \leq \gamma < \frac{3}{2}$.

2. Accuracy of the analytic approximation formula for the bond price in the one-factor interest rate model

In (1) the authors proposed an approximate analytical formula for the bond price in a one-factor interest rate model. They considered a model having a form (1) under the risk-neutral measure. It corresponds to the real measure process:

$$\mathrm{d}r = (\alpha + \beta r + \lambda(t,r)\sigma r^{\gamma})\,\mathrm{d}t + \sigma r^{\gamma}\mathrm{d}w$$

where $\lambda(t,r)$ is the so called market price of risk. For a general market price of risk function $\lambda(t,r)$, the price P of a zero-coupon bond can be obtained from a solution to the following partial differential equation:

$$-\partial_\tau P + \frac{1}{2}\sigma^2 r^{2\gamma}\partial_r^2 P + (\alpha + \beta r)\partial_r P - rP = 0,\ r > 0,\ \tau \in (0,T) \tag{2}$$

satisfying the initial condition $P(0,r) = 1$ for all $r > 0$ (see e.g. (4, Chapter 7)).

Definition 1. *By a complete solution to (2) we mean a function $P = P(\tau,r)$ having continuous partial derivatives $\partial_\tau P$, $\partial_r P$, $\partial_r^2 P$ on $Q_T = [0,\infty) \times (0,T)$, satisfying equation (2) on Q_T, the initial condition for $r \in [0,\infty)$ and fulfilling the following growth conditions: $|P(\tau,r)| \leq Me^{-mr^\delta}$ and $|P_r(\tau,r)| \leq M$ for any $r > 0, t \in (0,T)$, where $M, m, \delta > 0$ are constants.*

It is worth to note that comparison of approximate and exact solutions is meaningful only if the uniqueness of the exact solution is guaranteed. The next theorem gives us the uniqueness of a solution to (2) satisfying Definition 1. In order not to