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Two-Level Schwarz Preconditioners for Super Penalty Discontinuous Galerkin Methods

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Abstract. We extend the construction and analysis of the non-overlapping Schwarz preconditioners proposed in [2,3] to the (non-consistent) super penalty discontinuous Galerkin methods introduced in [5] and [8]. We show that the resulting preconditioners are scalable, and we provide the convergence estimates. We also present numerical experiments confirming the sharpness of the theoretical results.

AMS subject classifications: 65F10, 65N55, 65N30

Key words: Schwarz preconditioners, super penalty discontinuous Galerkin methods.

1 Introduction

Discontinuous Galerkin (DG) finite element methods have experienced a huge development in recent years. Although they have proved to enjoy many advantages in a number of circumstances, their practical utility is still limited by the much larger number of degrees of freedom they require compared to other classical discretization methods. To handle this possible limitation, some domain decomposition preconditioners have been proposed and analyzed in the past five years for *strongly consistent* and *stable* DG approximations of second order elliptic problems (cf. [2,3,12]).

In this paper we turn our attention to the non-consistent *super penalty* DG methods, namely the Babuška-Zlámal [5] and the Brezzi *et al.* [8] formulations. Although the idea of over-penalizing goes back to the early stage of the development of DG methods, this idea,

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together with the design of efficient solvers for the resulting schemes, have recently received a renewed interest (cf. [6,7]). Because of a non-consistency in the Babuška-Zlámal and Brezzi *et al.* formulations, a super penalty procedure has to be applied in order to achieve optimal approximation properties. The over-penalization has dramatic effects on the condition number of the resulting linear system of equations. In fact, if on a given *quasi uniform* mesh $\mathcal{T}_h$ with granularity h, polynomials of degrees ℓ_h are used for the approximation, the condition number of the resulting stiffness matrix is of order $\mathcal{O}(h^{-2\ell_h-2})$ (cf. [10]). In [2], it was numerically observed that the proposed non-overlapping Schwarz methods applied to the super penalty DG approximations result in a dramatic reduction on the condition number of the preconditioned linear systems of equations. However, the observed convergence rates differ considerably with respect to the ones exhibited by consistent DG discretizations. In this paper, we present the theoretical analysis that justifies those observed rates. We follow the theory developed in [2, 12] but using the natural norm for the super penalty schemes; *i.e.*, the norm induced by the bilinear form defining the scheme which does not scale as the energy norm of stable and consistent DG methods. As a consequence, some auxiliary results required in our analysis need to be reformulated and extended. The sharpness of our theoretical results is confirmed by some numerical experiments.

2 Super penalty discontinuous Galerkin discretizations

In this section, we set up some notation, introduce the model problem we will consider, and recall the variational formulation of super penalty DG methods. Throughout the paper, we shall use standard notation for Sobolev spaces (cf. [1]), and $x \lesssim y$ will mean that there exists a generic constant $C>0$ (that may not be the same at different occurrences but is always mesh independent) so that $x \leq Cy$.

Let $\Omega \subset \mathbb{R}^d$, $d=2,3$, be a convex bounded Lipschitz polygonal or polyhedral domain and $f \in L^2(\Omega)$. To ease the presentation, we consider the following model (toy) problem

$$-\Delta u = f \quad \text{in}\, \Omega\,, \quad u=0 \quad \text{on}\, \partial\Omega\,. \tag{2.1}$$

Meshes. Let $\mathcal{T}_h$ be a *shape-regular* and *quasi-uniform* conforming partition of the domain Ω into disjoint open elements T, where each T is the affine image of a fixed master element $\widehat{T}$, *i.e.*, $T=F_T(\widehat{T})$, and where $\widehat{T}$ is either the open unit d-simplex or the d-hypercube in $\mathbb{R}^d$, $d=2,3$. Letting h_T be the diameter of the element $T\in\mathcal{T}_h$, we define the mesh size h by $h=\max_{T\in\mathcal{T}_h} h_T$, and assume, for simplicity, that $h<1$. We denote by $\mathcal{F}_h^I$ and $\mathcal{F}_h^B$ the sets of all interior and boundary faces of $\mathcal{T}_h$, respectively, and set $\mathcal{F}_h=\mathcal{F}_h^I\cup\mathcal{F}_h^B$.

Remark 2.1. All the theory we present in this paper can be applied, with minor changes, to the case of non-matching grids, under suitable additional assumptions on $\mathcal{T}_h$; cf. [3].

Trace operators. Let $F\in\mathcal{F}_h^I$ be an interior face shared by two elements T^+ and T^- with outward normal unit vectors $\boldsymbol{n}^{\pm}$. For piecewise smooth vector-valued and scalar func-