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An Energy Regularization Method for the Backward Diffusion Problem and its Applications to Image Deblurring

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Abstract. For the backward diffusion equation, a stable discrete energy regularization algorithm is proposed. Existence and uniqueness of the numerical solution are given. Moreover, the error between the solution of the given backward diffusion equation and the numerical solution via the regularization method can be estimated. Some numerical experiments illustrate the efficiency of the method, and its application in image deblurring.

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1 Introduction

Let Ω be a bounded domain in $\mathbf{R}^n$ and let $\partial\Omega$ be its boundary. Then $\Sigma=\Omega\times(0,T)$ is a bounded domain in $\mathbf{R}^{n+1}$. We are interested in finding the numerical solution of the following backward diffusion problem:

$$\begin{aligned}
&\frac{\partial u}{\partial t}=\sum_{k,l=1}^{n}\frac{\partial}{\partial x_k}\left(a^{kl}(x)\frac{\partial u}{\partial x_l}\right)-c(x)u, \quad \text{in } \Sigma,\\
&u=0 \quad \left(or\,\frac{\partial u}{\partial \nu}=0\right), \quad \text{on } \partial\Omega\times[0,T),\\
&u(x,T)=g(x), \qquad x\in\Omega,
\end{aligned} \tag{1.1}$$

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where $c(x)$ is a given non-negative smooth function on $\overline{\Omega}$, $g(x)$ defines homogeneous boundary conditions on $\overline{\Omega}$, i.e.,

$$g(x)=0 \quad \text{or} \quad \frac{\partial g}{\partial \nu}=0 \quad \text{on } \partial\Omega. \tag{1.2}$$

Moreover,

$$\frac{\partial u}{\partial \nu}=\sum_{k,l=1}^{n} a^{kl}(x)\frac{\partial u}{\partial x_l}n_k, \tag{1.3}$$

where $\{n_k\}$ are the components of the unit normal vector on the boundary $\partial\Omega$ and $\{a^{kl}(x)\}$ is smooth on $\overline{\Omega}$ satisfying, for all $x\in\overline{\Omega}$,

$$\begin{aligned} &a^{kl}(x)=a^{lk}(x), \qquad 1\le k,\ l\le n,\\ &\alpha_0\sum_{k=1}^{n}\zeta_k^2\le\sum_{k,l=1}^{n}a^{kl}(x)\zeta_k\zeta_l\le\alpha_1\sum_{k=1}^{n}\zeta_k^2, \quad \forall\zeta=(\zeta_1,\cdots,\zeta_n)\in\mathbf{R}^n, \end{aligned} \tag{1.4}$$

where $0<\alpha_0<\alpha_1$ are two constants.

The problem (1.1) is reduced to the isotropic heat diffusion problem if we let $a^{kl}=c_0\delta_{kl}$, where c_0 is a positive constant and δ_{kl} is the Kronecker delta defined by

$$\delta_{kl}=\begin{cases} 1, & \text{when } k=l,\\ 0, & \text{when } k\neq l. \end{cases} \tag{1.5}$$

The backward diffusion problem (1.1) is a typical ill-posed problem in the sense of Hadamard [9,16]. The uniqueness of the given problem (1.1) can be found in [16], but the solution of problem (1.1) does not depend continuously on the given final data $g(x)$, and in general for any given function $g(x)$ with the vanishing boundary condition (1.2), there is no solution satisfying (1.1). In 1935, Tikhonov [1] obtained the backward diffusion problem by a geophysical interpretation, namely recovering the geothermal prehistory from contemporary data.

The problem (1.1) has been considered by many authors since the last century. After adding a priori information about the solution of the problem, such as smoothness or bounds on the solution in a given norm, we can restore stability and construct efficient numerical algorithms. Regularization methods are used by most authors to construct a solution of the ill-posed Cauchy problem for the backward diffusion equation. The main idea of most algorithms is solving a well-posed problem which is perturbed from the ill-posed one, and approximating the solution of the original problem with the solution of the well-posed one. A number of perturbations have been proposed, including the method of quasi-reversibility [3], pseudo-parabolic regularization [4], hyperbolic regularization [15]. Only the differential equation is perturbed in these methods. In [8], Showalter perturbed the initial condition rather than the differential equation, which has a better stability estimate than the previous ones.