INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 6, Number 3, Pages 505–519



# ANALYSIS OF A STABILIZED FINITE VOLUME METHOD FOR THE TRANSIENT STOKES EQUATIONS

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*This paper is dedicated to the special occasion of Professor Roland Glowinski's 70th birthdate.*

**Abstract.** This paper is concerned with the development and study of a stabilized finite volume method for the transient Stokes problem in two and three dimensions. The stabilization is based on two local Gauss integrals and is parameter-free. The analysis is based on a relationship between this new finite volume method and a stabilized finite element method using the lowest equal-order pair (i.e., the $P_1 - P_1$ pair). An error estimate of optimal order in the $H^1$-norm for velocity and an estimate in the $L^2$-norm for pressure are obtained. An optimal error estimate in the $L^2$-norm for the velocity is derived under an additional assumption on the body force.

**Key words.** Transient Stokes equations, stabilized finite volume method, *inf-sup* condition, local Gauss integrals, optimal error estimate, stability.

## 1. Introduction

Finite difference, finite element, and finite volume methods are three major numerical methods for solving engineering and science problems. The finite differences are easy to implement and locally conservative but not flexible to handle complex geometry. The finite elements have this flexibility but do not locally conserve mass. The finite volumes lie somewhere between the finite differences and the finite elements. They have the flexibility to handle complicated geometry, and their implementation capability is comparable to that of the finite differences. Moreover, their numerical solutions usually have certain conservation features that are desirable in many engineering and science applications.

The finite volume method has a variety of names: the control volume, covolume, and first-order generalized difference methods [3, 5, 7, 9, 12, 14, 22, 23, 24, 25, 29]. Compared to the finite element method, this method is harder to analyze; particularly, its stability and convergence for multidimensional partial differential equations is more difficult to establish. There exist some preliminary error estimates for second-order elliptic and parabolic partial differential problems. However, for more complex problems such as the Stokes problem under consideration, a fundamental stability and convergence theory for the finite volume method is limited.

Recently, a new stabilized finite element method based on two local Gauss integrals was developed for the stationary Stokes equations [18, 20]. This new method stabilizes the lowest equal-order (i.e., $P_1 - P_1$) elements by the residual of these local integrals on each triangular element. It is free of stabilization parameters, does not require any calculation of high-order derivatives or edge-based data structures, and can be implemented at the element level. Optimal error estimates were obtained using the technique of the standard finite element method [20]. More recently, this stabilized finite element method was extended to the finite volume method for the stationary Stokes equations [19]. After a relationship between this method and a stabilized finite element method was established, an error estimate of optimal order

Received by the editors August 30, 2008.
2000 *Mathematics Subject Classification.* 35Q10, 65N30, 76D05.

in the $L^2$- and $H^1$-norms for velocity and an estimate in the $L^2$-norm for pressure were obtained.

In this paper, we extend the definition and analysis of the stabilized finite volume method to the transient Stokes equations. The crucial argument in the analysis is how to use the relationship between the finite element and finite volume methods developed for the stationary problems to establish the desirable optimal error estimates for the transient problems. This crucial argument will be developed in detail here. This new finite volume method will be applied to porous media flow [6, 8].

This paper is organized as follows: In the next section, we introduce some notation, the transient Stokes equations, and their finite element discretizations. Then, in the third section, a stabilized finite volume method for the transient Stokes equations is developed, and a relationship between this method and a finite element method is considered. Stability and optimal order estimates for the finite volume method are obtained in the last three sections.

## 2. Preliminary

We focus on two dimensions; a generalization to three dimensions is straightforward. Let $\Omega$ be a bounded domain in $\Re^2$, with a Lipschitz-continuous boundary $\Gamma$, satisfying a further condition stated in (A1) below. The transient Stokes equations are

$$u_t - \nu\Delta u + \nabla p = f,\ \text{div}\ u = 0, \quad (x,t) \in \Omega \times (0,T], \tag{2.1}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \quad u(x,t)|_\Gamma = 0, \quad t \in [0,T], \tag{2.2}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t))$ represents the velocity vector, $p = p(x,t)$ the pressure, $f = f(x,t)$ the prescribed body force, $\nu > 0$ the viscosity, $T > 0$ the final time of interest, and $u_t = \partial u/\partial t$.

To introduce a variational formulation, set

$$X = (H_0^1(\Omega))^2,\ Y = (L^2(\Omega))^2,\ M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega); \int_\Omega q\ dx = 0 \right\},$$

$$V = \{v \in X : \text{div}\ v = 0\},\ D(A) = (H^2(\Omega))^2 \cap V.$$

As noted, a further assumption on $\Omega$ is needed:

(A1) Assume that $\Omega$ is regular in the sense that the unique solution $(v,q) \in (X,M)$ of the steady Stokes problem

$$-\Delta v + \nabla q = g, \qquad \text{div}\ v = 0 \quad \text{in } \Omega,\ v|_{\partial\Omega} = 0$$

for a prescribed $g \in Y$ exists and satisfies

$$\|v\|_2 + \|q\|_1 \le c\|g\|_0,$$

where $c > 0$ is a constant depending only on $\Omega$ and $\|\cdot\|_i$ denotes the usual norm of the Sobolev space $H^i(\Omega)$ or $(H^i(\Omega))^2$ for $i = 0,1,2$. Below the constant $c > 0$ will depend at most on the data $(\nu, T, u_0, \Omega)$.

We denote by $(\cdot,\cdot)$ and $\|\cdot\|_0$ the inner product and norm on $L^2(\Omega)$ or $(L^2(\Omega))^2$, as appropriate. The spaces $H_0^1(\Omega)$ and $X$ are equipped with their usual scalar product and norm

$$((u,v)) = (\nabla u, \nabla v),\ \|u\|_1 = ((u,u))^{1/2}.$$

Due to the norm equivalence between $\|u\|_1$ and $\|\nabla u\|_0$ on $H_0^1(\Omega)$, we are using the same notation for them: It is well known that for each $v \in X$ the following