COMMUNICATIONS IN COMPUTATIONAL PHYSICS
Vol. 3, No. 2, pp. 308-329

Commun. Comput. Phys.
February 2008

Efficient Solution of a Generalized Eigenvalue Problem Arising in a Thermoconvective Instability

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Received 9 January 2007; Accepted (in revised version) 25 March 2007

Available online 27 September 2007

Abstract. The aim of this paper is to develop an efficient numerical method to compute the eigenvalues of the stability analysis of a problem describing the motion of a fluid within a cylindrical container heated non-homogeneously from below. An axisymmetric stationary motion settles in, at certain values of the external parameters appearing in the set of partial differential equations modeling the problem. This basic solution is computed by discretizing the equations with a Chebyshev collocation method. Its linear stability is formulated with a generalized eigenvalue problem. The numerical approach (generalized Arnoldi method) uses the idea of preconditioning the eigenvalue problem with a modified Cayley transformation before applying the Arnoldi method. Previous works have dealt with transformations requiring regularity to one of the submatrices. In this article we extend those results to the case in which that submatrix is singular. This method allows a fast computation of the critical eigenvalues which determine whether the steady flow is stable or unstable. The algorithm based on this method is compared to the QZ method and is found to be computationally more efficient. The reliability of the computed eigenvalues in terms of stability is confirmed via pseudospectra calculations.

AMS subject classifications: 65F15, 35Q35

Key words: Rayleigh-Bénard convection, Chebyshev collocation, preconditioners, Arnoldi method, Hopf bifurcation.

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1 Introduction

The problem of thermoconvective instabilities in fluid layers driven by a temperature gradient has become a classical subject in fluid mechanics [1, 21]. It is well known that two different effects are responsible for the onset of motion when the temperature difference becomes larger than a certain threshold: gravity and capillary forces. The numerical resolution of these hydrodynamical problems has been widely studied [8, 10, 12]. In [13] the linear stability analysis of some convection problems is solved with a Chebyshev collocation method in the primitive variable formulation taking appropriate boundary conditions for pressure. In [15, 16] the method is applied to study a laterally heated fluid within an annulus. In this work we focus on a physical set-up that has been thoroughly studied in [20]. It consists of a fluid filling a cylindrical container. The upper surface is open to the air and the fluid is heated from below with a Gaussian temperature profile. The stationary and axisymmetric solution that appears at certain values of the external parameters, referred to as the basic state, is computed by using a Chebyshev collocation method as detailed in [20]. The linear stability analysis of these solutions is also formulated in its discrete version with a Chebyshev collocation method. The associated generalized eigenvalue problem has a novel matrix structure. The aim of this paper consists of describing an efficient numerical technique to compute the eigenvalues in this case. Pseudospectra are also calculated.

Many applications require the computation of eigenvalues in generalized eigenvalue problems. In particular these arise in the numerical study of linear stability of partial differential equations. There exist several methods to compute eigenvalues. For low dimensional matrices such as those appearing when collocation methods are used, the QZ algorithm is the standard method [11]. As this technique requires the computation of the whole set of eigenvalues, it is computationally very demanding. A more appropriate choice to treat these and even larger problems, are the selective algorithms which only compute the eigenvalues with largest real part, i.e., those closer to the instability threshold. The Jacobi-Davidson QZ method has been used in this context [26], but the most common algorithm of these is the Implicitly Restarted Arnoldi Method [17, 18, 24]. The application of this method to specific block structured matrices is treated in Ref. [5]. See [3, 22] for more recent exploration and application of these methods for finite elements in this context. The numerical approach implemented in this work uses the idea of preconditioning the eigenvalue problem with a modified Cayley transformation before applying the Arnoldi method. This idea has already been used in [5,23], but some results need to be proved before it can be extended to the resulting block structured matrices that appear with our collocation discretization. In particular if we express the problem as $Aw=\lambda Bw$, where

$$A=\begin{bmatrix} K & C \\ \widehat{C}^T & 0 \end{bmatrix}, \qquad B=\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}, \qquad w=\begin{bmatrix} v \\ p \end{bmatrix}, \tag{1.1}$$

the matrix M is singular whereas in the context of finite element approximation con-