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Enforcing the Discrete Maximum Principle for Linear Finite Element Solutions of Second-Order Elliptic Problems

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Abstract. The maximum principle is a basic qualitative property of the solution of second-order elliptic boundary value problems. The preservation of the qualitative characteristics, such as the maximum principle, in discrete model is one of the key requirements. It is well known that standard linear finite element solution does not satisfy maximum principle on general triangular meshes in 2D. In this paper we consider how to enforce discrete maximum principle for linear finite element solutions for the linear second-order self-adjoint elliptic equation. First approach is based on repair technique, which is a posteriori correction of the discrete solution. Second method is based on constrained optimization. Numerical tests that include anisotropic cases demonstrate how our method works for problems for which the standard finite element methods produce numerical solutions that violate the discrete maximum principle.

AMS subject classifications: 35J25, 65N99

Key words: Second-order elliptic problems, linear finite element solutions, discrete maximum principle, constrained optimization.

1 Introduction

In this paper we consider two approaches to enforce discrete maximum principle for linear finite element solution of the linear second-order self-adjoint elliptic equation without lower-order terms.

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It is well known that standard finite element methods can for some problems produce numerical solutions violating a discrete maximum principle (DMP) which is the discrete analog of the maximum principle, see, e.g., [1–7]. In the classical paper [8] Ciarlet and Raviart show that for the case of scalar isotropic diffusion coefficient the standard linear finite element method applied to Poisson equation satisfies the DMP on weakly acute triangular meshes. The weakly acute geometric condition is a typical condition under which some numerical methods produce solutions satisfying the DMP. The uniform constant anisotropic diffusion tensor can be transformed to the isotropic tensor (or the scalar diffusion coefficient) by rotating and scaling the coordinate system, so that one can use the acute conditions in the transformed coordinates. However, often one cannot choose the computational mesh or the anisotropy ratio is too big to provide a practical computational acute mesh in the transformed coordinates.

The issues related to the DMP have been studied by many researches. Here we try to review the recent contributions in the issues. The DMP for stationary heat conduction in nonlinear, inhomogeneous, and anisotropic media is analyzed by Krizek and Liu in [9, 10]. The dependence of DMP on mesh properties for finite element solutions of elliptic problems with mixed boundary conditions is considered by Karatson and Korotov in [11, 12]. Burman and Ern [13] have developed a nonlinear stabilized Galerkin approximation of the Laplace operator whose solutions satisfy the DMP without the need to satisfy the acute condition. However, this requires solving a nonlinear system of equations instead of a standard linear one. Le Potier has proposed a finite volume scheme for highly anisotropic diffusion problems on unstructured meshes [2] and improved it to the nonlinear version [3] which is monotone for a parabolic problem with sufficiently small time step. It has been further improved by Lipnikov et al. in [6], resulting in a nonlinear monotone finite volume scheme for elliptic problems which keeps positivity of the solution, however, can still violate the DMP. Mlacnik and Durlofsky [5] perform mesh optimization to improve the monotonicity of the numerical solution for highly anisotropic problems. A new mixed finite volume scheme for anisotropic diffusion problems has been developed by Droniou and Eymard in [4], however, it does not satisfy the DMP for highly anisotropic problems. The DMP has been investigated by means of discrete Green's function positivity by Draganescu et al. in [1]. The DMP for 1D problems with discontinuous coefficients is studied by Vejchodsky and Solin in [14]. The criteria for the monotonicity of control volume methods on quadrilateral meshes are derived by Nordbotten et al. in [7]. The elliptic solver on Cartesian grids for interface problems by Deng et al. [15] uses the standard scheme away from the interface, and a positive scheme at the interface is derived by using constrained optimization techniques. Hoteit et al. [16] study how to avoid violation of the DMP by the mixed-hybrid finite-element method (MHFEM) applied to a parabolic diffusion problem and propose two techniques reducing the MHFEM to finite difference methods obeying the DMP.

Our first approach to enforce discrete maximum principle is based on repair technique, [20–22], which is a posteriori correction of the discrete solution. Second method is based on constrained optimization. The quadratic optimization problem is related to