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QUINTIC SPLINE SOLUTIONS OF FOURTH ORDER BOUNDARY-VALUE PROBLEMS

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Dedicated to the memory of Dr. M. Rafique

Abstract. In this paper quintic spline is used for the numerical solutions of the fourth order linear special case boundary value problems. End conditions for the definition of spline are derived, consistent with the fourth order boundary value problem. The algorithm developed approximates the solutions, and their higher order derivatives. It has also been proved that the method is a second order convergent. Numerical illustrations are tabulated to demonstrate the practical usefulness of method.

Key Words. Quintic spline; Boundary value problem; End conditions.

1. Introduction

Spline functions are used in many areas such as interpolation, data fitting, numerical solution of ordinary and partial differential equations. Spline functions are also used in curve and surface designing.
Usmani [7], considered the fourth order boundary value problem to be the problem of bending a rectangular clamped beam of length l resting on an elastic foundation. The vertical deflection w of the beam satisfies the system

$$\left[L \; + \; \left(\frac{K}{D}\right)\right] w = D^{-1}q(x), \text{ where } L \equiv \frac{d^4}{dx^4} \,, \tag{1. 1}$$

$$w(0) = w(l) = w'(0) = w'(l) = 0, \tag{1. 2}$$

where D is the flexural rigidity of the beam, and k is the spring constant of the elastic foundation and the load $q(x)$ acts vertically downwards per unit length of the beam. Mathematically, the system (1.1) belongs to a general class of boundary value problems of the form

$$\left.\begin{array}{c} \left(\frac{d^4}{dx^4} \; + \; f(x)\right) y(x) \; = \; g(x), \; x \; \in \; [a,b], \\ y(a) = \alpha_0, \; y(b) \; = \; \alpha_1, \\ y'(a) = \gamma_0, \; y'(b) \; = \; \gamma_1, \end{array}\right\} \tag{1. 3}$$

where $\alpha_i, \; \gamma_i$; i= 0,1 are finite real constants and the functions $f(x)$ and $g(x)$ are continuous on [a,b]. The analytic solution of (1.3) for special choices of $f(x)$ and $g(x)$ are easily obtained, but for arbitrary choices, the analytic solution cannot be determined.

Numerical methods for obtaining an approximation to $y(x)$ are introduced. Usmani [7] derived numerical techniques of order 2, 4 and 6 for solution of a fourth

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order linear boundary value problem. Usmani [8] derived cubic, quartic, quintic and sextic spline solution of nonlinear boundary value problems. Usmani and Sakai [9] developed a quartic spline for the approximation of the solution of third order linear (special case) two point boundary value problems involving third order linear differential equation.

Papamichael and Worsey [4] derived end conditions for cubic spline interpolation at equally spaced knots. Papamichael and Worsey [5] have developed a cubic spline method, similar to that proposed by Daniel and Swartz [3] for second order problems.

Siddiqi and Twizell [10-13] presented the solutions of 6, 8, 10 and 12 order linear boundary value problems, using the sixth, eighth, tenth and twelfth degree splines.

In this paper a quintic spline method is described for the solution of (1.3). The end conditions for quintic spline interpolation, at equally spaced knots are derived, which is discussed in the next section.

2. Quintic Spline

Let Q be a quintic spline defined on $[a,\ b]$ with equally spaced knots

$$x_i = a\ +\ ih,\quad i = 0, 1, 2, ..., k, \tag{2. 4}$$

where

$$h = \frac{b-a}{k}. \tag{2. 5}$$

Moreover, for $i = 0, 1, 2, \ldots,\ k$, taking

$$Q(x_i) = y_i\ ,\ Q^{(1)}(x_i) = m_i, \tag{2. 6}$$

$$Q^{(2)}(x_i) = M_i\ ,\ Q^{(3)}(x_i) = n_i \tag{2. 7}$$

and

$$Q^{(4)}(x_i) = N_i. \tag{2. 8}$$

Also, let $y(x)$ be the exact solution of the system (1.3) and y_i be an approximation to $y(x_i)$, obtained by the quintic spline $Q(x_i)$. It may be noted that the $Q_i(x),\ i = 1, 2, 3, \ldots, k$ can be defined on the interval $[x_{i-1}, x_i]$, integrating

$$Q_i^4(x) = \frac{1}{h}[N_{i-1}(x_i - x) + N_i(x - x_{i-1})], \tag{2. 9}$$

four times w.r.t. x, gives

$$Q_i(x) = \frac{1}{120h}[N_{i-1}(x_i - x)^5\ +\ N_i(x - x_{i-1})^5]\ +\ \frac{Ax^3}{6}\ +\ \frac{Bx^2}{2}\ +\ Cx\ +\ D. \tag{2. 10}$$

To calculate the constants of integrations, the following conditions are used:

$$\begin{aligned} Q_i(x_i) = y_i\ , &\qquad Q^{(2)}{}_i(x_i) = M_i, \\ Q_i(x_{i-1}) = y_{i-1}\ , &\qquad Q^{(2)}{}_i(x_{i-1}) = M_{i-1}. \end{aligned} \tag{2. 11}$$

The identities of quintic splines for the solution of (1.3) can be written as

$$\begin{aligned} & m_{i-2}\ +\ 26m_{i-1}\ +\ 66m_i\ +\ 26m_{i+1}\ +\ m_{i+2} \\ =\ & \frac{5}{h}\left[-y_{i-2} - 10y_{i-1} + 10y_{i+1} + y_{i+2}\right]; \\ & i = 2, 3, ..., k-2, \end{aligned} \tag{2. 12}$$