INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 5, Number 2, Pages 320–330

A POSTERIORI ERROR ESTIMATION FOR A DUAL MIXED FINITE ELEMENT APPROXIMATION OF NON–NEWTONIAN FLUID FLOW PROBLEMS

MOHAMED FARHLOUL AND ABDEL-MALEK ZINE

Abstract. A dual mixed finite element method, for quasi–Newtonian fluid flow obeying to the power law, is constructed and analyzed in [8]. This mixed formulation possesses local (i.e., at element level) conservation properties (conservation of the momentum and the mass) as in the finite volume methods. We propose here an *a posteriori* error analysis for this mixed formulation.

Key Words. mixed finite element method, quasi–Newtonian fluid flow, *a posteriori* error analysis.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^2$ with a Lipschitz boundary Γ. Given $\boldsymbol{f}$, $\eta_0 > 0$ and r a real constant verifying $1 < r < \infty$, we consider the following boundary value problem : Find $(\boldsymbol{u}, p)$ such that

$$
(1) \qquad \begin{array}{ll}
-2\eta_0\, \boldsymbol{div}\left(|\boldsymbol{d}(\boldsymbol{u})|^{r-2}\, \boldsymbol{d}(\boldsymbol{u})\right) + \nabla p = \boldsymbol{f} & \text{in } \Omega \\
div\, \boldsymbol{u} = 0 & \text{in } \Omega \\
\boldsymbol{u} = 0 & \text{on } \Gamma
\end{array}
$$

where $\boldsymbol{d}(\boldsymbol{u})$ is the rate of strain tensor, $\boldsymbol{d}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^t)$, $\nabla \boldsymbol{u}$ is the tensor gradient of $\boldsymbol{u}$.

Throughout $|\cdot|$ denotes the Euclidian matrix norm, that is for $\boldsymbol{\tau}$, a $d \times d$ real matrix, $|\boldsymbol{\tau}| := \left[\sum_{i,j=1}^{d} (\tau_{ij})^2\right]^{1/2}$. The above system models the steady isothermal flow of an incompressible quasi-Newtonian fluid, $\boldsymbol{f}$ denotes the body force, $\boldsymbol{u}$ the velocity and p the pressure.

The well–posedness of the above nonlinear problem and its standard finite element approximation are well established in Baranger–Najib [1]. Extentions and improvements on the error bounds of [1] have appeared in Sandri [11] and in Barrett–Liu [2, 3].

In the framework of standard finite element method, an *a posteriori* error analysis is developed in Sandri [12]. A mixed finite element method has been introduced and analyzed in Farhloul–Zine [8]. Due to the introduction of the Cauchy stress tensor as a new variable, this new formulation possesses local (i.e., at element level) conservation properties (conservation of the momentum and the mass) as in the finite volume methods. Furthermore, it allows the approximations of all the physical variables (stress, velocity and pressure).

Received by the editors November 11, 2006 and, in revised form, March 3, 2007.
2000 *Mathematics Subject Classification.* 65N30, 65N15, 76A05, 76M10.

The aim of this work is to give an *a posteriori* error estimates for the mixed formulation developed in [8]. In the next section we recall the mixed formulation developed in [8] and then we give the *a posteriori* error estimates in section 3.

2. Mixed formulations

For the ease of the presentation, we take $\eta_0 = \frac{1}{2}$. Introducing $\boldsymbol{\sigma} = |\boldsymbol{d}(\boldsymbol{u})|^{r-2}\,\boldsymbol{d}(\boldsymbol{u})$ the extra–stress tensor, and using the fact that

$$|\boldsymbol{\sigma}|^{r'-2}\,\boldsymbol{\sigma} = \boldsymbol{d}(\boldsymbol{u}), \text{ where } r' \text{ is the conjugate of } r, \ i.e., \ \frac{1}{r} + \frac{1}{r'} = 1$$

problem (1) can be formulated as

$$\begin{array}{ll} -\,\boldsymbol{div}(\boldsymbol{\sigma} - p\mathbb{I}) = \boldsymbol{f} & \text{in } \Omega \\ div\,\boldsymbol{u} = 0 & \text{in } \Omega \\ A(\boldsymbol{\sigma}) := |\boldsymbol{\sigma}|^{r'-2}\,\boldsymbol{\sigma} = \boldsymbol{d}(\boldsymbol{u}) & \text{in } \Omega \\ \boldsymbol{u} = 0 & \text{on } \Gamma \end{array} \tag{2}$$

where $\boldsymbol{f} \in [L^{r'}(\Omega)]^2$, $\mathbb{I}$ is the identity tensor and for a given tensor $\boldsymbol{\tau} = (\tau_{ij})_{1\le i,j\le 2}$, $(\boldsymbol{div}\,\boldsymbol{\tau})_i = \sum_{j=1}^{2} \frac{\partial \tau_{ij}}{\partial x_j}$.

Note that for all $(\boldsymbol{\tau}, q) \in [L^{r'}(\Omega)]^{2\times 2} \times L_0^{r'}(\Omega)$ such that $\boldsymbol{div}(\boldsymbol{\tau} - q\mathbb{I}) \in [L^{r'}(\Omega)]^2$, as $div\,\boldsymbol{u} = 0$, one has

$$(A(\boldsymbol{\sigma}), \boldsymbol{\tau}) = (\boldsymbol{d}(\boldsymbol{u}), \boldsymbol{\tau}) = -(\boldsymbol{div}(\boldsymbol{\tau} - q\mathbb{I}), \boldsymbol{u}) - (\boldsymbol{\omega}, \boldsymbol{\tau}),$$

where $\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} - \nabla \boldsymbol{u}^t) \in [L^r(\Omega)]^{2\times 2}$ is the vorticity tensor and

$$L_0^{r'}(\Omega) = \left\{ q \in L^{r'}(\Omega); \ \int_\Omega q = 0\,dx \right\}.$$

In order to derive the mixed formulation of problem (2), we define the following spaces

$$\begin{aligned} \Sigma &= \left\{ \underset{\sim}{\tau} = (\boldsymbol{\tau}, q) \in [L^{r'}(\Omega)]^{2\times 2} \times L_0^{r'}(\Omega); \ \boldsymbol{div}(\boldsymbol{\tau} - q\mathbb{I}) \in [L^{r'}(\Omega)]^2 \right\}, \\ M &= \left\{ \underset{\sim}{v} = (\boldsymbol{v}, \boldsymbol{\eta}) \in [L^r(\Omega)]^2 \times [L^r(\Omega)]^{2\times 2}; \ \ \boldsymbol{\eta} + \boldsymbol{\eta}^t = 0 \right\}, \end{aligned}$$

equipped with their respective norms:

$$\| \underset{\sim}{\tau} \|_\Sigma = \left(\|\boldsymbol{\tau}\|_{0,r',\Omega}^{r'} + \|q\|_{0,r',\Omega}^{r'} + \|\boldsymbol{div}(\boldsymbol{\tau} - q\mathbb{I})\|_{0,r',\Omega}^{r'} \right)^{\frac{1}{r'}}, \ \| \underset{\sim}{v} \|_M = \left(\|\boldsymbol{v}\|_{0,r,\Omega}^{r} + \|\boldsymbol{\eta}\|_{0,r,\Omega}^{r} \right)^{\frac{1}{r}}.$$

The mixed formulation of (2) reads as follows: Find $\underset{\sim}{\sigma} = (\boldsymbol{\sigma}, p) \in \Sigma$ and $\boldsymbol{u} \in M$ such that

$$\begin{array}{ll} (A(\boldsymbol{\sigma}), \boldsymbol{\tau}) + (\boldsymbol{div}(\boldsymbol{\tau} - q\mathbb{I}), \boldsymbol{u}) + (\boldsymbol{\tau}, \boldsymbol{\omega}) = 0 & \forall\, \underset{\sim}{\tau} = (\boldsymbol{\tau}, q) \in \Sigma, \\ (\boldsymbol{div}(\boldsymbol{\sigma} - p\mathbb{I}), \boldsymbol{v}) + (\boldsymbol{\sigma}, \boldsymbol{\eta}) + (\boldsymbol{f}, \boldsymbol{v}) = 0 & \forall\, \underset{\sim}{v} = (\boldsymbol{v}, \boldsymbol{\eta}) \in M. \end{array} \tag{3}$$

The results concerning the existence, uniqueness and stability condition of the solution of (3) are developed in Farhloul-Zine [8]. However, we recall some results obtained in [8] that we need in the following section.

Proposition 1. *There exists a positive constant β such that*

$$\inf_{\underset{\sim}{v} \in M} \sup_{\underset{\sim}{\tau} \in \Sigma} \frac{(\boldsymbol{div}(\boldsymbol{\tau} - q\mathbb{I}), \boldsymbol{v}) + (\boldsymbol{\tau}, \boldsymbol{\eta})}{\| \underset{\sim}{v} \|_M \| \underset{\sim}{\tau} \|_\Sigma} \ge \beta. \tag{4}$$