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EXPLICIT HERMITE INTERPOLATION POLYNOMIALS VIA THE CYCLE INDEX WITH APPLICATIONS

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Abstract. The cycle index polynomial of a symmetric group is a basic tool in combinatorics and especially in Pólya enumeration theory. It seems irrelevant to numerical analysis. Through Faá di Bruno's formula, cycle index is connected with numerical analysis. In this work, the Hermite interpolation polynomial is explicitly expressed in terms of cycle index. Applications in Gauss-Turán quadrature formula are also considered.

Key Words. symmetric group, cycle index polynomial, Faá di Bruno's formula, Bell's polynomial, Hermite interpolation polynomial, Gauss-Turán quadrature formula.

1. Introduction

The cycle index polynomial of a symmetric group is a basic tool in combinatorics and plays an important role in Pólya enumeration theory. It is seemingly irrelevant to numerical analysis. To our best knowledge, its applications in numerical analysis is not systematical and many of them are in its disguises, see e.g., [16, 17, 21]. Actually, it can have wide applications in numerical analysis. Using cycle index polynomial, [19] find a closed form solution for a nonlinear system of equations, a problem arising in constructing nonlinear best quadrature formulas for Sobolev classes [18]. A further development and the proof of [19] are contained in [20].

As is well known, Faá di Bruno's formula applies when explicit higher derivatives of a composite function are sought [4]. Bell's polynomial arises naturally in Faá di Bruno's formula. The former is closely related to the cycle index polynomial. Therefore, it turns up in problems where higher derivatives of a composite function or its variants are involved. Algebraic and combinatorial tools and techniques can be exploited in such problems, which make analysis and computations easily accessible.

Based on Faá di Bruno's formula and logarithmic differentiation, the Hermite interpolation polynomial is explicitly expressed in terms of cycle index in this paper. And so are the divided differences with multiplicity. To our best knowledge, these formulas are new. Applications of these formulas to Gauss-Turan quadrature formulas are also included.

This work focuses on explicit Hermite interpolation polynomials via the cycle index, aiming at stimulating more attention to applications of the cycle index in numerical analysis.

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2. Cycle index of symmetric group and Hermite interpolation polynomial

Throughout, let $[n] := \{1, 2, \dots, n\}$ and $\mathfrak{G}$ be a permutation group of degree n. For any permutation $\sigma \in \mathfrak{G}$ and $i \in [n]$, let $c_i(\sigma)$ be the number of cycles of length of i in σ. The key result of Pólya theory is an expression for the number of orbits in terms of the cycle index polynomial of $\mathfrak{G}$. This polynomial, in n variables, is defined as follows [4, 10].

Definition 2.1.

$$Z(\mathfrak{G}; x_1, x_2, \dots, x_n) := \frac{1}{|\mathfrak{G}|} \sum_{\sigma \in \mathfrak{G}} x_1^{c_1(\sigma)} x_2^{c_2(\sigma)} \cdots x_n^{c_n(\sigma)}, \tag{2.1}$$

where $|\mathfrak{G}|$ is the order of $\mathfrak{G}$, i.e., the number of its elements. If $\mathfrak{G}$ =symmetric group $\mathfrak{S}_n$ of degree n, then its cycle index polynomial is written as

$$Z_n(x_1, x_2, \dots, x_n) := Z(\mathfrak{S}_n; x_1, x_2, \dots, x_n). \tag{2.2}$$

The following lemma can be easily verified (cf. [20]).

Lemma 2.2. (Recurrence relation)

$$\begin{aligned} &Z_0 = 1, \\ &nZ_n(x_1, x_2, \dots, x_n) = \sum_{k=1}^{n} x_k Z_{n-k}(x_1, x_2, \dots, x_{n-k}),\ n \geq 1. \end{aligned} \tag{2.3}$$

Here are first few examples of cycle index

$$\begin{aligned} Z_1(x_1) &= x_1, \\ Z_2(x_1, x_2) &= \frac{1}{2}(x_1^2 + x_2), \\ Z_3(x_1, x_2, x_3) &= \frac{1}{6}(x_1^3 + 3x_1x_2 + 2x_3), \\ Z_4(x_1, x_2, x_3, x_4) &= \frac{1}{24}(x_1^4 + 6x_1^2x_2 + 3x_2^2 + 8x_1x_3 + 6x_4). \end{aligned}$$

For convenience, we write

$$Z_n(x_k) := Z_n(x_k \mid k \in [n]) := Z_n(x_1, x_2, \dots, x_n).$$

Related to cycle index is Bell's polynomial which arises naturally in explicit expressions for high-order derivatives of a composite function. The following is (exponential) complete Bell's polynomial

$$Y_n(x_k) := Y_n(x_k \mid k \in [n]) := n! Z_n\Big(\frac{x_k}{(k-1)!} \mid k \in [n]\Big), \tag{2.4}$$

which can also be expressed as the sum of exponential partial Bell's polynomials $B_{n,m}$

$$Y_n(x_k) = \sum_{m=1}^{n} B_{n,m}(x_k).$$

Here

$$B_{n,m}(x_k | k \in [n]) := \sum_{\substack{a_1+2a_2+\cdots+na_n=n \\ a_1+a_2+\cdots+a_n=m}} \frac{n!}{a_1!(1!)^{a_1} a_2!(2!)^{a_2} \cdots a_n!(n!)^{a_n}} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

Bell's polynomials appear in Faá di Bruno's formula which explicitly gives the high-order derivatives of the composite function $g \circ f$ of functions g and f [4].