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AN IMPROVED EXPLICIT SCHEME FOR AGE-DEPENDENT POPULATION MODELS WITH SPATIAL DIFFUSION

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Abstract. In this work we present three age-structured models with spatial dependence. We introduce an improved explicit method, namely Super-Time-Stepping (STS) developed for parabolic problems and we use its modification for the numerical treatment of our models. We explain how the acceleration scheme can be adapted to the age-dependent models. We prove convergence of the method in case of Dirichlet boundary conditions and we demonstrate the accuracy and the efficiency of the Modified STS comparing it with other numerical algorithms of same or higher order, namely the explicit, fully implicit and Crank-Nicolson standard schemes.

Key Words. population dynamics, age-dependence, linear diffusion, linear and nonlinear models, finite difference method, numerical acceleration, modified super-time-stepping

1. Introduction.

During the last years, when modeling how populations change in time, it has been common to take into account not only the age structure of the species, but also their distribution in space. In 1973 Gurtin introduced spatial spread in age-dependent populations [9]. Later on many other authors have investigated the analytical aspects of various age-structured models with linear or nonlinear diffusion (for instance [4], [5], [6], [12], [16], [17]). Concerning the numerical treatment of the models arising in the field of population dynamics, numerical methods for models including only age or space structure, have been studied extensively (see [20, 21, 22] and the references cited therein). Much less research work has been done on models that include both - age and space. In the works of Kim [10], Kim-Park [11] and Milner [18], nonlinear models with nonlinear diffusion are treated. They propose and analyze some mixed numerical algorithms combining finite difference methods along the characteristic lines and finite element methods in the spatial variables. In the case of linear fertility and mortality functions, Lopez and Trigiante [15] have developed a finite difference scheme for an age-dependent model with Dirichlet boundary conditions and linear population flux. Ayati [3] proposes a numerical method for a nonlinear model with nonlinear diffusion which allows the use of variable time steps and independent age and time discretization. In [19] Pelovska and Boyadzhiev show how an additional acceleration of the Modified STS scheme can be obtained.

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In the present paper we propose a variation of the standard explicit scheme for the heat equation adapted for solving age-dependent population models with linear spatial diffusion. One of the main ideas used in the article is that along characteristics in the age-time direction the models below can be viewed as parabolic differential equations. The super-time-stepping algorithm that we employ is an acceleration method for explicit schemes for parabolic problems. STS relaxes the condition of stability at the end of every time step that is imposed for the normal explicit scheme and demands stability at the end of every super-step, where a super-step consists of K sub-steps. It implies that we can take larger time steps and consequently the total number of steps is reduced speeding up the computations, compared with the standard explicit scheme. The intermediate steps (sub-steps) are chosen non-uniformly from a formula that is given later.

The organization of the paper is as follows. In Section 2 we present the continuous models and our assumptions on them. In Section 3 the Super-Time-Stepping algorithm for the heat equation is presented and we show the connection between the problems with and without age structure. In Section 4 we give a modified version of STS adapted to our problems and we discuss its implementation. The convergence of the method is proved in Section 5. In Section 6 we give computational examples illustrating the benefits of the numerical scheme and we analyze our results. In Section 7 we make a brief discussion on the different approaches to the models with Neumann boundary conditions and in Section 8 we make some final remarks.

2. The continuous models.

The model we consider is similar to the Lotka-McKendrick's problem [20], but involving also the spatial structure of the individuals [19]. Let $p(a,t,x)$ be the **density of the population** where $a \in [0,a_+]$ denotes age and a_+ is the **maximum age**; $t>0$ denotes time and $x \in (0,1)$ denotes **spatial position**, then we have the following system:

$$
(1) \qquad \begin{cases} 1)\, p_t + p_a + \mu(a)p = Dp_{xx},\ a \in [0,a_+],\ t>0,\ x \in (0,1) \\ 2)\, p(0,t,x) = \displaystyle\int_0^{a_+} \beta(a)p(a,t,x)\ \mathrm{d}a = B(t,x),\ t>0,\ x\in(0,1) \\ 3)\, p(a,0,x) = p_0(a,x),\ a\in[0,a_+],\ x\in(0,1) \\ 4)\, p(a,t,0) = p(a,t,1) = 0,\ a\in[0,a_+],\ t>0 \end{cases}
$$

The functions $\beta(a)$ and $\mu(a)$ represent the **age specific fertility** and the **age specific mortality respectively**; $p_0(a,x)$ is the **initial distribution**; D is the **dispersal modulus** being constant; $B(t,x)$ is the **birth rate**, which gives the total number of offspring in one time unit at position x. Since homogeneous Dirichlet conditions on the boundaries of the region (0,1) are considered as "extremely inhospitable" (see [6]), we consider the same model but with Neumann boundary conditions which are imposed to describe a population without immigration or emigration:

$$
(2) \qquad \begin{cases} 1)\, p_t + p_a + \mu(a)p = Dp_{xx},\ a \in [0,a_+],\ t>0,\ x \in (0,1) \\ 2)\, p(0,t,x) = \displaystyle\int_0^{a_+} \beta(a)p(a,t,x)\ \mathrm{d}a = B(t,x),\ t>0,\ x\in(0,1) \\ 3)\, p(a,0,x) = p_0(a,x),\ a\in[0,a_+],\ x\in(0,1) \\ 4)\, p_x(a,t,0) = p_x(a,t,1) = 0,\ a\in[0,a_+],\ t>0 \end{cases}
$$