INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 5, Number 3, Pages 491–515

A CHARACTERIZATION OF SINGULAR ELECTROMAGNETIC FIELDS BY AN INDUCTIVE APPROACH

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(Communicated by Jun Zou)

Abstract. In this article, we are interested in the mathematical modeling of singular electromagnetic fields, in a non-convex polyhedral domain. We first describe the local trace (i. e. defined on a face) of the normal derivative of an L^2 function, with L^2 Laplacian. Among other things, this allows us to describe dual singularities of the Laplace problem with homogeneous Neumann boundary condition. We then provide generalized integration by parts formulae for the Laplace, divergence and curl operators. With the help of these results, one can split electromagnetic fields into regular and singular parts, which are then characterized. We also study the particular case of divergence-free and curl-free fields, and provide non-orthogonal decompositions that are numerically computable.

Key Words. Maxwell's equations, singular geometries, polyhedral domains.

Introduction

When one solves boundary value problems in a bounded polyhedron Ω of $\mathbb{R}^3$ with a Lipschitz boundary, it is well-known that the presence of reentrant corners and/or edges on the boundary deteriorates the smoothness of the solution [30, 28]. This problem is all the more relevant since boundary value problems which arise in practice, are often posed in domains with a simple but non smooth geometry, such as three-dimensional polyhedra.

More specifically, consider Maxwell's equations with perfect conductor boundary conditions and right-hand sides in $L^2(\Omega)$. Then the electromagnetic field $(\mathcal{E}, \mathcal{H})$ always belongs to $H^1(\Omega)^6$ when Ω is convex[1]. On the other hand, it is only guaranteed that it belongs to $H^\sigma(\Omega)^6$, for any $\sigma < \sigma_{max}$, with $\sigma_{max} \in]1/2, 1[$, when Ω is non-convex (see for instance [24]). In the latter case, strong electromagnetic fields can occur, near the reentrant corners and/or edges. For practical applications, we refer for instance to [31]. Nevertheless, one can split (cf. [7]) the field into two parts: a regular one, which belongs to $H^1(\Omega)^6$, and a singular one. According to [28, 7, 12, 24], the subspace of regular fields is closed, so one can choose to define the singular fields by orthogonality. Other approaches are possible and useful, see [23].

In the same way, when solving a problem involving the Laplace operator with data in $L^2(\Omega)$, the solution is in $H^2(\Omega)$ when Ω is either a convex polyhedron or

Received by the editors May 18, 2007 and, in revised form, September 25, 2007.

[1]As proven in [28, p. 12], if Ω is convex, then its boundary is automatically Lipschitz.

a bounded domain with a smooth boundary. However, it is only guaranteed to be in $H^{1+s}(\Omega)$ for $s < s_{max}$, when Ω is a non-convex polyhedron (one can prove that $s_{max} = \sigma_{max}$, see Section 3). Grisvard showed in [28] that a solution of the Laplace operator can be decomposed into the sum of a regular part and a singular part, the latter being called a *primal singularity*. This decomposition is based on a decomposition of $L^2(\Omega)$ into the sum of the image space of the regular parts and its orthogonal (the latter is the space of *dual singularities*).

As it is well known, the singular part of the electromagnetic field is linked [7, 8, 10] to the primal singularities of the Laplace problem, respectively with

- homogeneous Dirichlet boundary condition for the electric field $\mathcal{E}$;
- homogeneous Neumann boundary condition for the magnetic field $\mathcal{H}$.

For a comprehensive theory on this topic, we refer the reader to the works of Birman and Solomyak [7, 8, 9, 10, 11]. Among other results, they proved a splitting of the space of electromagnetic fields into a two-term *simple* sum. First, the subspace of regular fields. Second, the subspace made of gradients of solutions to the Laplace problem.

During the 1990s, Costabel and Dauge [25, 19, 20, 22, 23] provided new insight into the characterizations of the singularities of the electromagnetic fields, called afterwards *electromagnetic singularities*. In the process, they proved very useful density results.

In [2], we first studied, for L^2 functions with L^2 Laplacians, a possible definition of the trace on the boundary. Actually, it was proven that it can be understood locally – face by face – with values in $H^{-1/2}$-like Sobolev spaces. This being clarified, we inferred a *generalized integration by parts* (gibp) formula. Finally, in [4], we were able to describe precisely the space of all divergence-free singular electric fields. Indeed, starting from the orthogonality relationship with regular fields, the gibp formula allowed us to build a suitable characterization. In the present article, we would like to extend the results first to the case of magnetic fields and second to the case of any electric field, by using the same three step procedure.

The article is organized as follows. We first introduce some notations and define the Sobolev spaces that we will use throughout this paper. In the following Section, we recall some definitions on local traces together with the resulting gibp formula for the Laplace problem with Dirichlet boundary condition. These results are then extended to the Laplace operator with Neumann boundary condition. In Section 3, we transpose (part of) these results to the electromagnetic fields, from which, in Section 4, we can prove characterizations of the singular electromagnetic fields. Section 5 is devoted to the study of the divergence-free case. Then, in Section 6, we relate the regular/singular fields to the vector and scalar potentials. We also give their characterizations, using for this *ad hoc* isomorphisms. Finally, in the last Section, we consider curl-free spaces, that allow us to define non-orthogonal but numerically useful decompositions of the electromagnetic fields.

1. Notations and functional spaces

Let Ω be a bounded open set of $\mathbb{R}^3$, with a Lipschitz polyhedral boundary $\partial\Omega$. For simplicity reasons (cf. Remark 3.1), we assume that Ω is simply connected and that $\partial\Omega$ is connected. The unit outward normal to $\partial\Omega$ is denoted by $\mathbf{n}$. We call