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AN A PRIORI ERROR ANALYSIS OF OPERATOR UPSCALING FOR THE ACOUSTIC WAVE EQUATION

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Abstract. In many earth science problems, the scales of interest range from centimeters to kilometers. Computer power and time limitations prevent inclusion of all the fine-scale features in most models. However, upscaling methods allow creation of physically realistic and computationally feasible models. Instead of solving the problem completely on the fine scale, upscaling methods produce a coarse-scale solution that includes some of the fine-scale detail. Operator-based upscaling applied to the pressure/acceleration formulation of the acoustic wave equation solves the problem via decomposition of the solution into coarse and subgrid pieces. To capture local fine-scale information, small subgrid problems are solved independently in each coarse block. Then these local subgrid solutions are included in the definition of the coarse problem. In this paper, accuracy of the upscaled solution is determined via a detailed finite element analysis of the continuous-in-time and fully-discrete two-scale numerical schemes. We use lowest-order Raviart-Thomas mixed finite element approximation spaces on both the coarse and fine scales. Energy techniques show that in the L^2 norm the upscaled acceleration converges linearly on the coarse scale, and pressure (which is not upscaled in this implementation) converges linearly on the fine scale. The fully discrete scheme is also shown to be second-order in time. Three numerical experiments confirm the theoretical rate of convergence results.

Key Words. upscaling, convergence analysis, acoustic wave propagation, multiscale methods, error estimates.

1. Introduction

Data required for deep crustal seismic studies, time-lapse seismology, detailed near-surface environmental cleanup applications, and other modern-day earth-science problems can easily run into the terabyte range or beyond. Further, depending on the questions which need to be addressed in a particular study, the data may span a range of scales from centimeters to kilometers. Over the past few decades computational scientists and geoscientists have contributed to the development of various methods aimed at obtaining accurate and cost-effective solutions to these modeling problems. Finite-difference methods have long been accepted as an easy-to-implement and relatively accurate way to solve the discrete wave equation [14], [27], [28], [33]. While finite element methods for solving the wave equation have been proposed, they have not been as widely embraced in the geoscience modeling

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community. These methods are better able to handle complex domain geometries, but they are more difficult to implement than finite-difference methods [28]. (For examples of finite element approaches to solving the wave equation, see [6], [7], [15], [16], [21], [22], [24], [25].)

Large-scale acoustic and elastic wave propagation in two and three dimensions has become computationally feasible in large part due to the successful implementation of data parallel algorithms (see [29] and works cited therein). In spatial (or data) parallelism, each processor has ownership of a subset of the total domain. The processor is responsible for allocating ghost cells at the boundaries of its subdomain, for updating the finite-difference solution over its portion of the domain (including along the ghost cells), and for communicating the boundary solution data to its immediate neighbors.

As an alternative to data parallelism (which includes all the collected data in the modeling but parcels the data out to different processors to reduce the computational load), techniques have emerged which attempt to either (1) determine the most important bits of data to incorporate in the model [34], (2) determine effective or homogenized input *parameters* for solving a coarser-scale problem [8], [11], [13], [23], or (3) approximate the *solution* via a subgrid upscaling procedure [1], [4], [12], [19].

In an earlier paper by Vdovina et al. [32], the authors presented the first extension of operator-based upscaling (originally developed for elliptic flow problems) to the acoustic wave equation. The key idea in operator upscaling is to solve the problem via a two-scale decomposition of the solution [4]. At the first stage, the problem is solved for the fine-scale (subgrid) unknowns defined locally within each coarse block. The method makes use of homogeneous Neumann boundary conditions between coarse blocks (in the first step) which allow for localization of the subgrid problems. At the second stage, we use the subgrid solutions to determine a new coarse-grid operator defined on the global domain. While the method was originally developed in the context of mixed finite elements, in practice the computationally intensive part of the algorithm (subgrid solve) is accomplished via finite differences. (We exploit the connection between lowest-order mixed finite elements and cell-centered finite differences [30], [32].) The method does not involve explicit averaging of the input parameters nor does it require scale separation or periodicity. We parallelize only the first stage of the algorithm (solving the much smaller coarse problem in serial). The main advantage of this upscaling approach over standard data parallelism is that there is no communication between processors since the subgrid problems decouple due to the boundary conditions used in the first stage of the process. We tested the numerical accuracy of the method on large domains with geophysically realistic input data typical of subsurface environments. Numerical experiments show that sub-wavelength scale heterogeneities were captured by the upscaled solution.

In a related paper, Korostyshevskaya and Minkoff [26] analyze the physical problem solved by the upscaling technique. What this analysis highlights is that the numerical upscaling process solves a constitutive equation similar in form to the original equation. The constitutive equation relates acceleration to the gradient of pressure. For the coarse (upscaled) problem, however, the parameter field (density) reduces to an averaged density along coarse block edges. Similarly, when analyzing the pressure equation, we find that the upscaled solution solves the original wave