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CONVERGENCE ANALYSIS OF SPECTRAL METHODS FOR INTEGRO-DIFFERENTIAL EQUATIONS WITH VANISHING PROPORTIONAL DELAYS*

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Abstract

We describe the application of the spectral method to delay integro-differential equations with proportional delays. It is shown that the resulting numerical solutions exhibit the spectral convergence order. Extensions to equations with more general (nonlinear) vanishing delays are also discussed.

Mathematics subject classification: 65R20, 34K28.
Key words: Delay integro-differential equations, Proportional delays, Spectral methods, Convergence analysis.

1. Introduction

We consider the delay integro-differential equation of the form

$$\begin{aligned} y'(t) =& a(t)y(t) + b(t)y(qt) + \int_0^t K_0(t-s)y(s)ds \\ &+ \int_0^{qt} K_1(t-s)y(s)ds + g(t), \quad t \in I := [0,T], \end{aligned} \tag{1.1a}$$

$$y(0) = y_0, \tag{1.1b}$$

where $0 < q < 1$, $a(t)$ and $b(t)$ are smooth functions on $I := [0,T]$ and K_0, $K_1 \in C(I)$. The special case corresponding to $K_0(t,s) \equiv 0$, $K_1(t,s) \equiv 0$, $g(t) = 0$, yields the (variable coefficient) *pantograph equation*. Results on the existence, uniqueness and regularity of solutions may be found in [3-6].

It has been shown in [6] that the approximation of the solution of (1.1) by collocation using piecewise polynomials of degree $m \geq 1$ and uniform meshes does not lead to the classical $\mathcal{O}(h^{2m})$- superconvergence at the mesh points when collocation is at the Gauss points; for $m \geq 2$ the optimal order is only $m+2$. Thus, it is of interest to investigate if the numerical solution of (1.1) by spectral methods leads to a higher (exponential) convergence order.

It will be shown that the results on the exponential order of convergence of the spectral method for the pantograph DDE [7] and for Volterra type integral equations [11, 12] remain valid for pantograph-type integro-differential equation (1.1).

In Section 2 we describe the spectral method for the integro-delay differential equation. This is followed, in Section 3, by corresponding results on the attainable order of convergence of these spectral methods and by remarks (Section 4) on their extension to equations with nonlinear vanishing delays. Section 5 is used to illustrate the convergence results by numerical examples.

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2. Spectral Method

Let $\{t_k\}_{k=0}^N$ be the set of the $(N+1)$ Gauss-Legendre, or Gauss-Radau, or Gauss-Lobatto points in $[-1,1]$ and denote by $\mathcal{P}_N$ the space of real polynomials of degree not exceeding N. Integrating (1.1a) from $[0,t_i]$ gives

$$\begin{aligned} y(t_i) =& y_0 + \int_0^{t_i} a(s)y(s)ds + \int_0^{t_i} b(s)y(qs)ds + \int_0^{t_i}\left(\int_0^s K_0(s-v)y(v)dv\right)ds \\ &+ \int_0^{t_i}\left(\int_0^{qs} K_1(s-v)y(v)dv\right)ds + \int_0^{t_i} g(s)ds. \end{aligned} \tag{2.1}$$

We will describe and analyzed spectral methods on the standard interval $[-1,1]$. Hence using for t_i $(i=1,\cdots,N)$ the linear transformation $s=\frac{t_i}{2}\theta+\frac{t_i}{2}$, we get

$$\begin{aligned} y(t_i) =& y_0 + \frac{t_i}{2}\int_{-1}^1 a\Big(\frac{t_i}{2}(\theta+1)\Big)y\Big(\frac{t_i}{2}(\theta+1)\Big)d\theta + \frac{t_i}{2}\int_{-1}^1 b\Big(\frac{t_i}{2}(\theta+1)\Big)y\Big(\frac{qt_i}{2}(\theta+1)\Big)d\theta \\ &+ \frac{t_i}{2}\int_{-1}^1\left(\int_0^{\frac{t_i}{2}(\theta+1)} K_0\Big(\frac{t_i}{2}(\theta+1)-v\Big)y(v)dv\right)d\theta \\ &+ \frac{t_i}{2}\int_{-1}^1\left(\int_0^{\frac{qt_i}{2}(\theta+1)} K_1\Big(\frac{t_i}{2}(\theta+1)-v\Big)y(v)dv\right)d\theta + G(t_i), \end{aligned} \tag{2.2}$$

where

$$G(t_i) := \int_0^{t_i} g(s)ds.$$

Using the $(N+1)$-point Gauss-Legendre, or Gauss-Radau, or Gauss-Lobatto quadrature formula relative to the Legendre weight leads to the semi-discretised spectral equations

$$\begin{aligned} y(t_i) \approx& y_0 + \frac{t_i}{2}\sum_{k=0}^N a(\tau_{ik})y(\tau_{ik})\omega_k + \frac{t_i}{2}\sum_{k=0}^N b(\tau_{ik})y(q\tau_{ik})\omega_k \\ &+ \frac{t_i}{2}\sum_{k=0}^N\left(\int_0^{\tau_{ik}} K_0(\tau_{ik}-v)y(v)dv\right)\omega_k \\ &+ \frac{t_i}{2}\sum_{k=0}^N\left(\int_0^{q\tau_{ik}} K_1(\tau_{ik}-v)y(v)dv\right)\omega_k + G(t_i), \end{aligned} \tag{2.3}$$

which we rewrite in the form

$$\begin{aligned} y(t_i) \approx& y_0 + \frac{t_i}{2}\sum_{k=0}^N a(\tau_{ik})y(\tau_{ik})\omega_k + \frac{t_i}{2}\sum_{k=0}^N b(\tau_{ik})y(q\tau_{ik})\omega_k \\ &+ \frac{t_i}{2}\sum_{k=0}^N\left(\frac{\tau_{ik}}{2}\int_{-1}^1 K_0\Big(\tau_{ik}-\frac{\tau_{ik}}{2}(\theta+1)\Big)y\Big(\frac{\tau_{ik}}{2}(\theta+1)\Big)d\theta\right)\omega_k \\ &+ \frac{t_i}{2}\sum_{k=0}^N\left(\frac{q\tau_{ik}}{2}\int_{-1}^1 K_1\Big(\tau_{ik}-\frac{q\tau_{ik}}{2}(\theta+1)\Big)y\Big(\frac{q\tau_{ik}}{2}(\theta+1)\Big)d\theta\right)\omega_k + G(t_i), \end{aligned} \tag{2.4}$$

where $\tau_{ik} := \frac{t_i}{2}(\theta_k+1)$ and $i=0,\cdots,N$.