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A Method of Lines Based on Immersed Finite Elements for Parabolic Moving Interface Problems

Tao Lin[1], Yanping Lin[2,3,*] and Xu Zhang[1]

[1] *Department of Mathematics, Virginia Tech, Blacksburg, VA 24061, USA*
[2] *Department of Applied Mathematics, Hong Kong Polytechnic University, Hung Hom, Hong Kong*
[3] *Department of Mathematical and Statistics Science, University of Alberta, Edmonton AB, T6G 2G1, Canada*

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Dedicated to Graeme Fairweather on the occasion of his 70^{th} birthday.

Abstract. This article extends the finite element method of lines to a parabolic initial boundary value problem whose diffusion coefficient is discontinuous across an interface that changes with respect to time. The method presented here uses immersed finite element (IFE) functions for the discretization in spatial variables that can be carried out over a fixed mesh (such as a Cartesian mesh if desired), and this feature makes it possible to reduce the parabolic equation to a system of ordinary differential equations (ODE) through the usual semi-discretization procedure. Therefore, with a suitable choice of the ODE solver, this method can reliably and efficiently solve a parabolic moving interface problem over a fixed structured (Cartesian) mesh. Numerical examples are presented to demonstrate features of this new method.

AMS subject classifications: 35R05, 65M20, 65M60

Key words: Immersed finite element, moving interface, method of lines, Cartesian mesh.

1 Introduction

In this article, we consider the following parabolic moving interface problem:

$$u_t - \nabla\cdot(\beta\nabla u) = f(t,X), \qquad \text{if } X\in\Omega, \qquad t\in(0,T_{end}], \tag{1.1a}$$
$$u(t,X) = g(t,X), \qquad \text{if } X\in\partial\Omega, \qquad t\in(0,T_{end}], \tag{1.1b}$$
$$u(0,X) = u_0(X), \qquad \text{if } X\in\overline{\Omega}, \tag{1.1c}$$

*Corresponding author.
Email: tlin@math.vt.edu (T. Lin), malin@polyu.edu.hk (Y. Lin), xuz@vt.edu (X. Zhang)

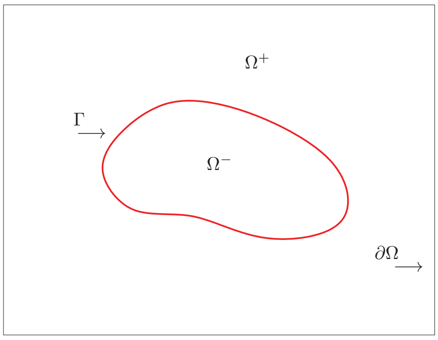

Figure 1: A sketch of the domain for the moving interface problem.

where the domain $\Omega \subset \mathbb{R}^2$ is assumed to be an open rectangle (or a union of open rectangles) that is separated into two sub-domains $\Omega^+(t)$ and $\Omega^-(t)$ by a curve $\Gamma(t)$ defined by a smooth function $\Gamma : [0, T_{end}] \to \Omega$, see Fig. 1 for an illustration of the solution domain Ω. The diffusion coefficient $\beta(t,X)$ is discontinuous across the interface $\Gamma(t)$. For simplicity's sake, we assume that $\beta(t,X)$ is a piece-wise constant function defined as follows:

$$\beta(t,X) = \begin{cases} \beta^-, & \text{if } X \in \Omega^-(t), \\ \beta^+, & \text{if } X \in \Omega^+(t). \end{cases} \tag{1.2}$$

Across the moving interface $\Gamma(t)$, the solution $u(t,X)$ is required to satisfy the usual jump conditions:

$$[u]|_{\Gamma(t)} = 0, \tag{1.3a}$$

$$[\beta \nabla u \cdot \mathbf{n}]|_{\Gamma(t)} = 0. \tag{1.3b}$$

The moving interface problem described by (1.1a)-(1.3b) appears in many applications, such as field injection problems [14, 15, 33, 35, 42] and Stefan problems [9, 34]. The two-phase Stefan problem consists of this kind of parabolic moving interface problem and an ordinary differential equation based on the physics for tracking the interface location between the two material phases. In this article, we are focusing on the difficulties in solving the parabolic initial boundary value problem with an evolving interface and hope that the method presented here can be extended to more complicated problems.

Conventional finite element (FE) methods can solve the parabolic differential equations (PDEs) satisfactorily [38]. In dealing with interface problems, if the interface does not change its shape and location, then methods such as those discussed in [38] can be straightforwardly utilized provided that the meshes are tailored to match the interface [2, 7, 10]; otherwise, their convergence might be impaired [5]. We call meshes of this type as body-fitting meshes, in which each element is essentially on one side of the interface, see the plot on the left in Fig. 2 for an illustration.

However, the requirement of using body-fitting mesh makes traditional FE methods inefficient for solving moving interface problems. First, for a problem with a moving