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REDUCED BASIS METHOD FOR PARAMETRIZED ELLIPTIC ADVECTION–REACTION PROBLEMS*

Luca Dedè
MOX–Modeling & Scientific Computing, Dipartimento di Matematica "F. Brioschi", Politecnico di Milano, P.zza Leonardo da Vinci 32, 20133, Milano, Italy
Email: luca.dede@polimi.it

Abstract

In this work we consider the Reduced Basis method for the solution of parametrized advection–reaction partial differential equations. For the generation of the basis we adopt a stabilized finite element method and we define the Reduced Basis method in the "primal–dual" formulation for this stabilized problem. We provide a priori Reduced Basis error estimates and we discuss the effects of the finite element approximation on the Reduced Basis error. We propose an adaptive algorithm, based on the a posteriori Reduced Basis error estimate, for the selection of the sample sets upon which the basis are built; the idea leading this algorithm is the minimization of the computational costs associated with the solution of the Reduced Basis problem. Numerical tests demonstrate the efficiency, in terms of computational costs, of the "primal–dual" Reduced Basis approach with respect to an "only primal" one.

Mathematics subject classification: 35J25, 35L50, 65N15, 65N30, 76R99.
Key words: Parametrized advection–reaction partial differential equations, Reduced Basis method, "primal–dual" reduced basis approach, Stabilized finite element method, a posteriori error estimation.

1. Introduction

The Reduced Basis (RB) method is a computational approach which allows rapid and reliable predictions of functional outputs associated with the solution of Partial Differential Equations (PDEs) with parametric dependence [1,6,13–18]. Indeed, the RB method has a wide range of relevant applications in the characterization of engineering components or systems which require the prediction of certain "quantities of interest", e.g., fluid dynamics, heat and mass transfer problems, (see, e.g., [11, 13, 20, 25, 27, 30]), as well as linear elasticity applications (see, e.g., [7, 12, 13, 26]) and many other physical problems (see, e.g., [3, 16, 18, 28]). Environmental problems represent a promising field of application for the RB method. Preliminary investigations have been made in [19, 21] for pollution problems in air, for which the RB method has been adopted to evaluate the concentration of pollutants emitted by industrial sites in certain zones of observation, such as cities [5], and to speed up the solution of the associated optimal control problems. Parametrized steady advection–diffusion PDEs have been used in this context with both geometrical and physical parameters, such as the location of industrial plants, the intensity or direction of the wind field and the diffusion coefficient.

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In this work we investigate the RB method for the evaluation of outputs, dependent on the solution of parametrized advection–reaction PDEs, in view of environmental applications, for which diffusion phenomena are negligible w.r.t. the transport and reaction ones.

The RB method is based on the decoupling of the generation and the projection stages of the approximation procedures, which leads to a decoupled offline–online computational approach. The complexity of the offline step, in which the basis are generated, depends on the dimension of the "truth" space, let say N_t, to which belongs the "truth" solution for a given parameter. The complexity of the online stage depends on the dimension of the RB space, let say N, with $N \ll N_t$, and on the parametric dependence.

For the definition of the "truth" space, we use the Finite Element (FE) method [23]. In order to get rid of the numerical instabilities due to the transport term of the hyperbolic advection–reaction PDE, we use the Streamline Diffusion Finite Element (SDFE) stabilized method [23, 31]. This leads to the transformation of the original hyperbolic PDE into a new one, with elliptic nature. We define the RB method for this parametrized stabilized advection–reaction problem, for which the affine decomposition property holds, and we consider the "primal–dual" RB approach [13,16,28], which requires the definition of a dual problem. This approach is well–suited both for the approximation and the error evaluation of the output and, as we highlight in this work, also for the reduction of the computational costs associated with the RB online step w.r.t. those of the "only primal" RB approach (without the dual problem). We provide a priori RB error estimates for both the solution and the output, thus highlighting the role of the FE approximation and stabilization in the RB method, being the total error composed by both the FE and RB ones. In particular, we show that, for the problem under consideration, the "complexity" of the RB approximation increases as the FE one improves by reducing the mesh size. We also report for this problem the a posteriori RB error estimate for the output according to [16,28]. We remark that the idea of using stabilized FE for the definition of the "truth" space has been already introduced in [19] for the solution of optimal control problems, even if a priori and a posteriori RB estimates and an error analysis for the FE and RB approximations have not been discussed. We use an adaptive algorithm for the definition of the RB basis, which is led by the a posteriori RB error estimate and based on a criterium of minimization of the online computational costs. Two numerical tests, inspired by environmental problems, are discussed; moreover, we experimentally show that the RB approximation is stable, if the FE one is stable.

This work is organized as follows. In Sec.2 we introduce the parametrized advection–reaction PDEs in an abstract setting and two problems with physical and geometrical parameters. In Sec.3 we provide the FE approximation of the parametrized problem, after having introduced the stabilization by means of the SDFE method; an a priori error analysis is reported for a particular case. Sec.4 deals with the RB method, for which the "primal–dual" RB approach is considered. Both a priori and a posteriori RB estimates are provided and the proposed adaptive algorithm for the choice of the sample sets is outlined. In Sec.5 we report some considerations about the numerical solution of the parametrized advection–reaction PDEs by means of the FE and RB methods. We discuss in Sec.6 two numerical tests. Concluding remarks follow.

2. Parametrized Advection–Reaction Equations

We introduce in an abstract setting the parametrized advection–reaction PDEs and we specify two problems with physical and geometrical parameters.