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A BINARY LEVEL SET MODEL FOR ELLIPTIC INVERSE PROBLEMS WITH DISCONTINUOUS COEFFICIENTS

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Abstract. In this paper we propose a variant of a binary level set approach for solving elliptic problems with piecewise constant coefficients. The inverse problem is solved by a variational augmented Lagrangian approach with a total variation regularisation. In the binary formulation, the seeked interfaces between the domains with different values of the coefficient are represented by discontinuities of the level set functions. The level set functions shall only take two discrete values, i.e. 1 and -1, but the minimisation functional is smooth. Our formulation can, under moderate amount of noise in the observations, recover rather complicated geometries without requiring any initial curves of the geometries, only a reasonable guess of the constant levels is needed. Numerical results show that our implementation of this formulation has a faster convergence than the traditional level set formulation used on the same problems.

Key Words. Inverse problems, parameter identification, elliptic equation, augmented Lagrangian optimisation, level set methods, total variation regularisation.

1. Introduction

Consider the elliptic partial differential equation with Dirichlet boundary conditions:

$$\begin{aligned} -\nabla \cdot (q(\mathbf{x})\nabla u) &= f \qquad \text{in } \Omega \subset \mathbb{R}^2 \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

In this paper we will use observations of the function u to recover the coefficient $q(\mathbf{x})$ by approximating it with a piecewise constant function. The function f is assumed known, and $\partial\Omega$ is the boundary of our domain Ω. This problem is a model problem for many real applications, for example, reservoir simulations [18], medical imaging [14, 16] and underground water investigations [11]. Even as a purely academic problem, this model problem has turned out to be rather difficult to solve numerically.

The problem of recovering the geometry of the coefficient discontinuities has motivated a number of approaches in the literature [8, 9, 12, 13]. A proper regularisation is often applied to control the jumps and the geometry of the discontinuities, see for example [8, 9]. Several approaches have also been used to represent the coefficient implicitly, especially a number of level set methods have been proposed for this purpose; see [6, 10, 20, 27, 17, 1, 2, 3].

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For representing $q(\mathbf{x})$ we apply a piecewise constant formulation of the level set method. The original level set method was proposed by Osher and Sethian [25] for tracing interfaces between different phases of fluid flow. It has later been a versatile tool for representing and tracking interfaces separating a domain into subdomains. The method has been applied in a wide range of applications, i.e. reservoir simulations, inverse problems, image analysis, and optimal shape design problems. For recently surveys of level set methods see [30, 7, 24].

The representation of the interfaces in the standard level set formulation is done implicitly by the zero level set of one or several functions. The corresponding Euler-Lagrange equations give the evolution equations for the level set functions and the different constant values of the coefficient. In these methods the level set functions are forced to be signed distance functions, and are therefore the solution of Hamilton-Jacobi equations.

The *binary level set method*, which we apply in this paper, is classified as a piecewise constant level set approach. For this method, the level set functions are discontinuous functions at convergence, and should only take a fixed number of predefined constant values. Hence, they should not be distance functions as in the continuous formulation. In the binary method, we require the convergence values of the level set functions to be -1 or 1. When solving the optimisation problem with this imposed requirement, we need to minimise a smooth convex functional under a quadratic constraint. The level set method applied in this paper is similar to the method proposed in [21] used for image segmentation. This idea has also appeared in some earlier work [19, 29] for image segmentation. The binary level set idea is in fact very similar to the phase field model applied for many phase transition problems [5, 26, 4]. In this work, we use a novel way to treat the requirement that the level set functions should take the values ± 1.

Other related works, utilising piecewise constant level set functions, are presented in [22, 28]. The method in [22] are within a similar type of framework as the method presented in this paper. A difference is that for the approach in [22] just one level set function is required to identify an arbitrary number of phases, while in the binary method a combination of several level set functions is utilised to represent multiple regions. The method in [28] is based on an optimality condition for the final curves, and this method does, contrary to the other methods, not require any solutions of PDEs.

In [10], Chan and Tai have performed a study on the elliptic inverse problem which is closely related to the work presented in this paper. They use continuous level set functions in a more standard level set formulation. As in their approach, we will in this work formulate the method in a variational setting, and apply an augmented Lagrangian approach for solving the minimisation problem. The Euler-Lagrange equations give the evolution equations for the level set functions.

Since the minimisation problem is highly ill-posed, we need to regularise the problem. The regularisation applied here is the total variation norm of the recovered coefficient. This will indirectly control both the length of the level set curves and the jumps in the coefficients, see [10, 31].

The contribution in this paper is the use of binary represented level set functions for solving the elliptic inverse problem. In our implementation, the relation between the coefficients and the level set functions is constructed such that it to a large extent can reduce the ill-posedness of the inverse problem, and at the same time be able to reconstruct rather complicated geometries.