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PIECEWISE CONSTANT LEVEL SET METHOD FOR MULTIPHASE MOTION

HONGWEI LI AND XUE-CHENG TAI

Abstract. We apply the Piecewise Constant Level Set Method (PCLSM) to a multiphase motion problem, especially the pure mean curvature motion. We use one level set function to represent multiple regions, and by associating an energy functional which consists of surface tension (proportional to length), we formulate a variational approach for the mean curvature motion problem. Some operator-splitting schemes are used to solve the problem efficiently. Numerical experiments are supplied to show the efficiency for motion by mean curvature for multiphase problems.

Key Words. multiphase motion, level set method, piecewise constant, operator splitting

1. Introduction

Several approaches for multiphase motion have been developed, such as front tracking methods [4], Monte-Carlo methods, phase field methods, diffusion generated MBO-like method and variational level set approach etc, see [37, 24, 22, 23, 18]. Usually, the model problem involves curves meeting at a point with prescribed angles, See Fig.(1). Each interface Γ_{ij}, separates regions Ω_i and Ω_j and moves with a normal velocity

$$v_{ij} = f_{ij}\kappa_{ij} + (e_i - e_j). \tag{1}$$

where κ_{ij} is the local curvature, f_{ij} is the constant surface tension of Γ_{ij}, and e_i corresponds to the bulk energy. This model problem can be obtained by associating an energy functional E to the motion, which involves the length of each interface and the area of each subregion, i.e.

$$\begin{aligned} E &= E_1 + E_2 \\ E_1 &= \sum_{1\leqslant i<j\leqslant n} f_{ij} Length(\Gamma_{ij}) \\ E_2 &= \sum_{1\leqslant i\leqslant n} e_i Area(\Omega_i). \end{aligned} \tag{2}$$

By minimizing this energy functional, the internal interfaces are driven to equilibrium. Examples of such kind of motion include solid, liquid, grain or multiphase boundaries [37].

Since complicated topological changes can occur for the model problem, front tracking methods are generally hard to implement, especially for more than two dimensions. Monte-Carlo methods are usually too slow to find good approximations. Phase field methods may also have difficulties to resolve the interface layers [24].

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MBO method was proposed by Merriman, Bence and Osher [18]. This method naturally handles complicated topological changes with junctions. MBO method is based on the diffusion of characteristic functions of each region, followed by a certain reassignment step. And it has been shown that MBO is appropriate for pure mean curvature flow, in which the bulk energies are zero and the f_{ij} are all equal to the same positive constant. A rigorous convergence proof for two phase mean curvature motion has been given in [3].

Our work is especially inspired by [18] and [37]. In [37], a variational approach and a practical method was proposed for treating junctions even when topological mergings and breakings occur. Their method also limits angles to the classical condition [33]

$$\frac{\sin\theta_1}{f_{23}} = \frac{\sin\theta_2}{f_{31}} = \frac{\sin\theta_3}{f_{12}}, \tag{3}$$

at triple junction points.

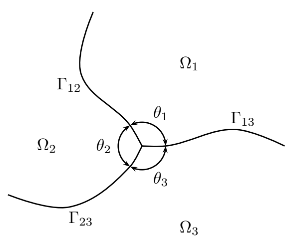

FIGURE 1. The interfaces Γ_{ij}, subjected to angles θ_1,θ_2,θ_3

In addition to [18] and [37], the traditional level set idea of Osher and Sethian [19] has been used for motion by mean curvature in [14, 6, 8, 26, 29, 5, 20]. For traditional level set methods, one needs to reinitialize the level set function to be a signed distance function during the iterations, and cautions must be taken with respect to the discretization of Heaviside and Dirac functions [7]. The piecewise constant level set method (PCLSM) [13, 12, 11] is trying to use some alternatives for the traditional level set idea, and one doesn't need to care about these issues [13]. Moreover, the formulation is truly variational and just one level set function is needed to identify arbitrary number of phases [34, 37]. Another advantage is that the constraint and energy functionals for our approach are all smooth functionals which do not involve Heaviside and Dirac functions.

In this paper, we apply the piecewise constant level set method [11, 13, 12] to multiple phase motion problems. The remainder of this paper is organized as follows: In section 2, we introduce the piecewise constant level set method (PCLSM), and formulate the mean curvature motion problem into a variational problem by PCLSM. In section 3, two algorithms will be proposed using some operator-splitting techniques. Section 4 goes for numerical experiments.

2. The piecewise constant level set formulation

We shall use the PCLSM of [13] to solve the mean curvature motion problem. The essential idea of the PCLSM of [13] is to use a piecewise constant level set function to identify the subdomains. Assume that we need to partition the domain