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# INTERVAL-BASED REDUCED-ORDER MODELS FOR UNSTEADY FLUID FLOW

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*This paper is dedicated to Max Gunzburger on the occasion of his 60th birthday*

**Abstract.** A number of practical engineering problems require the repeated simulation of unsteady fluid flows. These problems include the control, optimization and uncertainty quantification of fluid systems. To make many of these problems tractable, reduced-order modeling has been used to minimize the simulation requirements. For nonlinear, time-dependent problems, such as the Navier-Stokes equations, reduced-order models are typically based on the proper orthogonal decomposition (POD) combined with Galerkin projection. We study several modifications to this reduced-order modeling approach motivated by the optimization problem underlying POD. Our discussion centers on a method known as the principal interval decomposition (PID) due to IJzerman.

**Key Words.** reduced-order modeling, proper orthogonal decomposition, principal interval decomposition, surrogate model, optimization.

## 1. Introduction

The use of reduced-order modeling in control and optimization has led to practical solutions for extremely challenging problems, such as control of high-Reynolds number flow [29], solutions to the Hamilton-Jacobi-Bellman equation arising in nonlinear feedback control [21], and design of materials for desired microstructure-sensitive material properties [10]. The development of accurate and reliable reduced-order models is critical to the success of these solution approaches. In this paper, we discuss reduced-order models for unsteady flows and suggest a number of potential improvements.

As in the examples above, reduced-order modeling for nonlinear, time-dependent problems typically consists of a basis selection strategy coupled with a model building step. This usually involves a proper orthogonal decomposition (POD) [22] of simulation time snapshots followed by Galerkin projection to build the model (cf. [13] and [1]). The POD and its variants are also known as Karhunen-Loève expansions [19, 22], principal component analysis (PCA) [14], and empirical orthogonal functions (EOF) [23] among others. This method of coupling a reduced-basis with Galerkin projection to build reduced-order models of fluid flow has developed over the past two decades [28] as more complex simulation [26] and control [17] applications emerged. However, reduced-order modeling remains both a “science” and

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an "art." There are examples where POD combined with Galerkin projection can produce unstable models from stable, linear systems [33]. Thus, there is an industry in constructing new approaches for reduced-order modeling of both linear and nonlinear systems.

A number of improvements to the basis selection and model building have been suggested over the past two decades. Improvements to the Galerkin model-building portion include the theoretically promising, nonlinear Galerkin methods [24] have not proven to be a practical alternative to the standard Galerkin methods [18, 20]. However, for flow problems, a number of specialized approaches have shown promise in managing the energy decay in the model. These are based on modifying or tuning the viscosity term in the model [6, 9, 31].

To develop adequate basis functions from POD, a (number of) representative simulation(s) (known as an input collection) is needed. This frequently requires input from a disciplinary specialist although a number of heuristics based on the spectral content of the input collection have been suggested, eg. [8]. As well shall see, the standard application of POD can miss flow structures that are dynamically relevant but are only expressed for small time intervals. Problems with a convective nature (eg. travelling waves) produce POD bases that don't capture any problem solution structure. This can prevent adequate dimension reduction.

Recent approaches consider the simultaneous calculation of the basis and the model [32, 2]. In this paper, we consider the little known principal interval decomposition (PID) [16] and some new extensions. The PID simultaneously finds a basis element and the time interval over which it is expressed in the data. Thus, it is well suited for convective problems. In a number of natural extensions to the PID, we consider multiple basis elements per time interval as well as comment on practical criteria for choosing the length of the time interval. The result of this last modification is the development of an *a priori* estimate of the error associated with the resulting PID/Galerkin model. Numerical experiments involving the unsteady von Karman vortex shedding past a square cylinder demonstrate the effectiveness of these modifications.

## 2. Overview of reduced-order modeling

We begin by giving an overview of the POD/Galerkin framework for reduced-order modeling in this section. This will provide the context and notation to explain the PID approaches in the next section. To facilitate this overview, consider the (two-dimensional, incompressible) Navier-Stokes equations

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nabla\cdot\boldsymbol{\tau}(\mathbf{u}) + \mathbf{f} \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{2}$$

where $\mathbf{u} = (u, v)$ is the velocity vector, $p$ is the pressure, $\boldsymbol{\tau}(\mathbf{u}) = 2\nu\boldsymbol{\varepsilon}(\mathbf{u})$ is the deviatoric fluid stress tensor, $\nu$ is the kinematic viscosity, and $\boldsymbol{\varepsilon}(\mathbf{u}) = 1/2\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$ is the symmetric strain-rate tensor. With suitable nondimensionalization, $\nu$ is the reciprocal of the Reynolds number.

An important step in reduced-order modeling is to find a suitable set of basis functions. We will give an overview of methods to find them below. For now, assume we have a reduced-basis of dimension $r$, $\{\boldsymbol{\phi}_j(\cdot)\}_{j=1}^r$ with $\boldsymbol{\phi}_j(\cdot) \in \left[H^1(\Omega)\right]^2$ where $\Omega$ is the flow domain. Using this basis, we represent our reduced-order approximation