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REDUCED ORDER MODELING OF SOME NONLINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. Determining accurate statistical information about outputs from ensembles of realizations is not generally possible whenever the input-output map involves the (computational) solution of systems of nonlinear partial differential equations (PDEs). This is due to the high cost of effecting each realization. Recently, in applications such as control and optimization that also require multiple solutions of PDEs, there has been much interest in reduced-order models (ROMs) that greatly reduce the cost of determining approximate solutions. We explore the use of ROMs for determining outputs that depend on solutions of stochastic PDEs. One is then able to cheaply determine much larger ensembles, but this increase in sample size is countered by the lower fidelity of the ROM used to approximate the state. In the contexts of proper orthogonal decomposition-based ROMs, we explore these counteracting effects on the accuracy of statistical information about outputs determined from ensembles of solutions.

Key Words. reduced order modeling, stochastic differential equations, brownian motion, monte carlo methods, finite element methods.

1. Introduction

Realistic simulations of complex systems governed by nonlinear partial differential equations must account for the "noisy" features of the modeled phenomena, such as material properties, coefficients, domain geometry, excitations and boundary data. "Noise" can be understood as uncertainties in the specification of the physical model; because of noise, the behavior of a complex system is at least partially unpredictable. A simulation can attempt to capture the noisy aspects of a system by describing the simulation input data as random fields. This turns the problem into a stochastic partial differential equation (SPDE). We will consider such problems, characterized by nonlinear partial differential equations, and for which the input data are not purely deterministic; for example, the coefficients or the right-hand-side of the partial differential equation may be regarded as sums of a deterministic and stochastic function.

For a given system, various stochastic perturbation techniques have been considered [1, 2, 4–6, 16, 19, 36, 64, 65]. This paper will focus on nonlinear SPDE's in which the stochastic inputs are modeled as white noise, i.e., they are not significantly correlated. The aim of our work is to efficiently determine statistical information

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about the random field $u = u(t, \mathbf{x}; \omega)$ from numerical approximations of the nonlinear SPDE driven by white noise:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au - \gamma N(u) + g + \epsilon \frac{\mathrm{d}\mathbb{W}}{\mathrm{d}t}, \quad \mathbf{x} \in \mathcal{D}, \quad \omega \in \Omega, \quad t > 0. \tag{1.1}$$

Here $\mathcal{D} \subset \mathbb{R}^N$ is a convex, bounded and polygonal spatial domain, $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space described in section 2, and A is a linear second-order elliptic operator with deterministic coefficients, defined on a space of functions satisfying certain boundary conditions, $N(u)$ is a nonlinear function of the random process u, g represents a deterministic function and $\mathbb{W}$ denotes an infinite dimensional *Brownian motion* or *Wiener process*. The additive noise that appears in (1.1) is in the form of space-time Brownian white noise as described in section 2.1. The amplitudes of the noise and the nonlinearity are controlled by parameters ϵ and γ, respectively. Once the equation is reformulated into a weak form, the usual Galerkin finite element approach can be used to produce a discretized system suitable for solution on a computer.

Generally, obtaining precise statistics about ensembles of realizations of nonlinear SPDEs such as (1.1) entails a high cost in both memory and CPU. This cost is exhibited in many recent attempts on similar problems [9, 13, 26, 27]. Even with the use of reliable nonlinear solvers and carefully chosen solution schemes, these computations involve formidable work. Typical finite element codes may require the use of many thousands of degrees of freedom for the accurate simulation of deterministic PDEs. The situation becomes far worse when the same techniques are extended to SPDEs [20] for which multiple realizations are usually required.

It is natural to consider a reduced-order model (ROM), such as [10, 11]. A reduced-order model attempts to determine acceptable approximate solutions of a PDE while using very few degrees of freedom. One way to achieve this efficiency is for the models to use basis functions that are in some way intimately connected to the problem being solved. Once a low-dimensional reduced basis has been determined, it may be used in a new Galerkin system to solve related instances of the PDE. In this way, a ROM may be used to efficiently explore the behavior of large ensembles of PDE solutions. This is the kind of efficiency needed when attempting to compute realistic statistics from outputs of the SPDE.

There have been many reduced-order modeling techniques proposed; see [10, 11, 33, 37] and the references cited therein. The most popular reduced-order modeling approach for nonlinear PDEs is based on proper orthogonal decomposition (POD) analysis. POD begins with a set of $\widetilde{m}$ precomputed solutions of the equation, often called *snapshots*; these could be generated by evaluating the computational solution of a transient problem at many instants of time or over a range of values of the problem parameters. These solutions are presumably obtained using costly, large-scale, high-fidelity codes. The K-dimensional POD basis is then formed from the K eigenvectors corresponding to the dominant eigenvalues of the snapshot correlation matrix. This basis may then be used to construct a new finite element system of much reduced order, suitable for generating approximate solutions, at least within a limited range of the underlying snapshot data. POD-based model reduction has been applied with some success to several problems, most notably in fluid mechanics. For detailed discussions, one may consult [3, 7, 8, 10–12, 17, 25, 28, 29, 32–34, 43–48, 50–54, 59, 60, 62, 63].

The efficiency of a POD basis comes from its low dimension combined with its good approximating power. However, the ability of a POD-based basis to approximate the state of a system is totally dependent on the information contained in the