MULTIDIMENSIONAL RELAXATION APPROXIMATIONS FOR HYPERBOLIC SYSTEMS OF CONSERVATION LAWS [*1)]

Mohammed Seaïd

(*Universität Kaiserslautern Fachbereich Mathematik, 67663 Kaiserslautern, Germany*
Email: seaid@mathematik.uni-kl.de)

Abstract

We construct and implement a non-oscillatory relaxation scheme for multidimensional hyperbolic systems of conservation laws. The method transforms the nonlinear hyperbolic system to a semilinear model with a relaxation source term and linear characteristics which can be solved numerically without using either Riemann solver or linear iterations. To discretize the relaxation system we consider a high-resolution reconstruction in space and a TVD Runge-Kutta time integration. Detailed formulation of the scheme is given for problems in three space dimensions and numerical experiments are implemented in both scalar and system cases to show the effectiveness of the method.

Mathematics subject classification: 35L60, 35L65, 82B40, 65M20, 74S10, 65L06.
Key words: Multidimensional hyperbolic systems, Relaxation methods, Non-oscillatory reconstructions, Asymptotic-preserving schemes.

1. Introduction

In this paper we are interested in solving numerically the multidimensional hyperbolic system of conservation laws

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{U})}{\partial y} + \frac{\partial \mathbf{H}(\mathbf{U})}{\partial z} = \mathbf{0}, \quad t > 0, \quad (x, y, z) \in \mathbb{R}^3, \tag{1.1a}$$

$$\mathbf{U}(t = 0, x, y, z) = \mathbf{U}_0(x, y, z), \tag{1.1b}$$

where $\mathbf{U}(t, x, y, z) \in \mathbb{R}^N$ is a vector of conserved quantities; $\mathbf{F}(\mathbf{U}) \in \mathbb{R}^N$, $\mathbf{G}(\mathbf{U}) \in \mathbb{R}^N$ and $\mathbf{H}(\mathbf{U}) \in \mathbb{R}^N$ are nonlinear flux functions; and $\mathbf{U}_0 \in \mathbb{R}^N$ is given initial data. We assume that the Jacobian matrices $\partial \mathbf{F}/\partial \mathbf{U}$, $\partial \mathbf{G}/\partial \mathbf{U}$ and $\partial \mathbf{H}/\partial \mathbf{U}$ are diagonalizable with real eigenvalues $\{\lambda_1, \cdots, \lambda_N\}$, $\{\mu_1, \cdots, \mu_N\}$ and $\{\xi_1, \cdots, \xi_N\}$, respectively.

The relaxation system proposed in [7] and considered in this paper reads

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{V}}{\partial x} + \frac{\partial \mathbf{W}}{\partial y} + \frac{\partial \mathbf{Z}}{\partial z} = \mathbf{0}, \tag{1.2a}$$

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} = -\frac{1}{\varepsilon}\big(\mathbf{V} - \mathbf{F}(\mathbf{U})\big), \tag{1.2b}$$

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{B}\frac{\partial \mathbf{U}}{\partial y} = -\frac{1}{\varepsilon}\big(\mathbf{W} - \mathbf{G}(\mathbf{U})\big), \tag{1.2c}$$

$$\frac{\partial \mathbf{Z}}{\partial t} + \mathbf{C}\frac{\partial \mathbf{U}}{\partial z} = -\frac{1}{\varepsilon}\big(\mathbf{Z} - \mathbf{H}(\mathbf{U})\big), \tag{1.2d}$$

where $\mathbf{V} \in \mathbb{R}^N$, $\mathbf{W} \in \mathbb{R}^N$ and $\mathbf{Z} \in \mathbb{R}^N$ are relaxation variables; $\mathbf{A} = \mathrm{diag}\{A_1, \cdots, A_N\}$, $\mathbf{B} = \mathrm{diag}\{B_1, \cdots, B_N\}$ and $\mathbf{C} = \mathrm{diag}\{C_1, \cdots, C_N\}$ are positive diagonal matrices; and $\varepsilon > 0$ is the relaxation time. The relaxation system (1.2) has a typical semilinear structure with linear characteristic variables defined by

$$\mathbf{V} \pm \sqrt{\mathbf{A}}\mathbf{U}, \qquad \mathbf{W} \pm \sqrt{\mathbf{B}}\mathbf{U} \qquad \text{and} \qquad \mathbf{Z} \pm \sqrt{\mathbf{C}}\mathbf{U}. \tag{1.3}$$

Formally, in the zero relaxation limit $\varepsilon \longrightarrow 0$, we recover the original system (1.1) provided the subcharacteristic condition [7, 11, 8],

$$\frac{\lambda_\nu^2}{A_\nu} + \frac{\mu_\nu^2}{B_\nu} + \frac{\xi_\nu^2}{C_\nu} \leq 1, \qquad \forall \quad \nu = 1, \cdots, N, \tag{1.4}$$

holds in (1.2). Note that if we project the relaxation variables into the local equilibrium

$$\mathbf{V} = \mathbf{F}(\mathbf{U}), \quad \mathbf{W} = \mathbf{G}(\mathbf{U}) \quad \text{and} \quad \mathbf{Z} = \mathbf{H}(\mathbf{U}), \tag{1.5}$$

then the first equation of (1.2) reduces to the original conservation laws (1.1). Further references on the analysis and convergence of relaxation methods can be found in [19, 20] among others.

Our aim in this paper is to reconstruct high order relaxation schemes for the hyperbolic systems (1.1) in multi-space dimensions. The central key for such reconstruction is the combination of Weighted Essentially Non-Oscillatory (WENO) polynomials for the space discretization and asymptotic-preserving implicit-explicit (IMEX) methods for the time integration. Although we concentrate on a third-order reconstruction, the formalism presented here can readily be applied to develop relaxation methods with arbitrary order of accuracy. It is worthwhile to mention that WENO schemes of order between 7 and 11 can be found in the literature, compare [3] for details. We also should mention that central schemes [6] and non-uniform mesh methods [18] offer useful numerical tools for solving multi-dimensional hyperbolic systems of conservation laws. However, all these methods solve the relaxed original problem (1.1) instead of the relaxation system (1.2). In many practical applications one may also be interested in the transient regimes before the equilibrium. For instance, there is strong links between relaxation methods and kinetic or lattice Boltzmann schemes used in the frame of Boltzmann equation (as a simple case, the BGK model which offers a similar structure as the one we referred to as relaxation system (1.2) with ε denotes the Knudsen number). It is well known that, by keeping the Knudsen number small one can derive inviscid Euler problems from the kinetic equation. However, it is a challenging problem to construct consistent space and time discretization of the transient equations that preserve the asymptotic limit and converge to the correct numerical solution of the limit equations as the Knudsen number goes to zero. Therefore, one of the purposes of our work was to combine in a formal way a high order space and time discretizations in order to construct a numerical method that works for both small and large relaxation rates ε.

The design of high order relaxation schemes has been partially addressed by other authors in [13, 10], however, in those references the formulations and numerical results are given only for the one-dimensional problems. Extension to the two-dimensional hyperbolic systems was recently discussed in [14, 5] along with a comparison between relaxed schemes ($\varepsilon = 0$) and the well-established central methods. Relaxation methods were also used in [15] for the shallow water equations in one and two space dimensions and in [16] for the two-dimensional Riemann problems in gas dynamics. To our knowledge, this is the first time that multi-dimensional hyperbolic systems of conservation laws are approximated by relaxation techniques. To demonstrate the basic algorithms, and show that it can adapt to multi-dimensional features of a solution,