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SUPERCONVERGENCE PROPERTIES OF DISCONTINUOUS GALERKIN METHODS FOR TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract. Three discontinuous Galerkin methods (SIPG, NIPG, DG) are considered for solving a one-dimensional elliptic problem. Superconvergence for the error at the interior node points and the derivative of the error at Gauss points are considered. All theorectical results obtained in the paper are supported by the results of numerical experiments.

Key Words. Discontinuous Galerkin methods, superconvergence, 1D problem.

1. Introduction

Discontinuous Galerkin (DG) methods are effective numerical methods for solving differential equations. DG methods dates back to the early 1970s when Nitche [21] introduced the concept of replacing the Lagrange multiplier used in hybrid formulations with averaged normal fluxes at the boundaries and added stabilization terms to produce optimal convergence rates. Early work on DG methods can be found in Reed and Hill [25], Percell and Wheeler [23], Arnold [1], Delves and Hall [14], etc. DG methods have been developed and analyzed for both hyperbolic and elliptic problems in parallel. There are two types of DG methods: one is in primal formulations and another is in mixed formulations. Both formulation may or may not include interior penalty terms. We refer to Chen [11] for a review of relationships on different DG methods for solving second order elliptic differential equations. Several papers have been published for rigorous a priori error estimates of DG methods. See, for examples, the paper by Arnold, Brezzi, Cockburn and Martin [2] for DG methods in primal formulations and Castillo, Cockburn, Perugia and Schötzau [7] for DG methods in mixed formulations.

Superconvergence in finite element methods have been studied for several decades. We refer to Krizek and Neittaanmaki [18], Wahlbin [28], Lin and Zhu [32] and the literature cited there. However, there are only a few papers dealing with superconvergence for discontinuous Galerkin methods. In Cockburn, Kanschat, Perugia and Schötau [12] and Castillo, Cockburn, Schötzau and Schwab [8], some superconvergence results have been obtained for the local discontinuous Galerkin method. There are no superconvergence results reported in the literature about the discontinuous Galerkin method in its primal formulation (the local DG method is in mixed formulation). It is the aim of this paper to study the superconvergence property of discontinuous Galerkin methods in non-mixed formulation. We will study a simple one-dimensional problem and analyze the superconvergence property of

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two types of DG methods. In the first type of DG methods we will study interior penalty Galerkin (IPG) methods; This includes the symmetric interior penalty (SIPG) method and the non-symmetric interior penalty (NIPG) method. In the second type of DG methods we consider a DG method without penalty terms. This method was developed by Baumann etc. in [3]. The main results obtained in this paper are the following: 1) superconvergence for the derivative of the error at Gauss points for SIPG and NIPG methods when super-penalty is used in the DG formulations; 2) superconvergence for the derivative of the error at Gauss points for all three methods when the mesh is uniform and the degree of the polynomials of finite element space is an odd number; 3) superconvergence for the averaged errors at node points for the SIPG and NIPG methods using at least piecewise quadratical polynomials.

The superconvergence property for DG methods solving partial differential equations is currently under investigation.

The paper is organized in the following way: In section 2, we derive the weak formulation used for discontinuous Galerkin methods, introduce some notation and define the interpolation operator; In section 3, superconvergence results are derived for SIPG and NIPG methods; The corresponding results for a DG method without penalty is in Section 4. Finally the numerical experiments that support our theoretical results are presented in Section 5.

2. Preliminaries

For simplicity, we consider the following two-point boundary value problem with mixed boundary conditions:

$$
\begin{cases}
-(p(x)u'(x))' = f(x), \quad x \in (0,1), \\
u(x) = u_d(x) \text{ at } x \in \Gamma_D, \\
u'(x) = u_n(x) \text{ at } x \in \Gamma_N,
\end{cases}
\tag{2.1}
$$

where coefficients $p(x)$ and $f(x)$ are assumed to be sufficiently smooth and satisfy

$$
p(x) \geq p_0 > 0, \qquad \text{for all } x \in (0,1).
\tag{2.2}
$$

And $\Gamma_D \subset \{0,1\}$, $\Gamma_N \subset \{0,1\}$ are the sets of points where boundary conditions are defined and satisfy $\Gamma_D \cap \Gamma_N = \emptyset$, $\Gamma_D \cup \Gamma_N = \{0,1\}$, $\Gamma_D \neq \emptyset$.

To formulate the discontinuous Galerkin method for solving (2.1), we divide Ω into N subintervals:

$$
0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1, \qquad \text{with } h_i = x_i - x_{i-1}, \quad h = \max_{1\leq i\leq N} h_i,
$$

and assume that the partition is quasi uniform in the sense that

$$
h \leq C \min_{1\leq i\leq N} h_i.
$$

Here, and throughout the paper, letter C denotes a generic constant independent of the mesh size h and the functions u, v, etc. Let $\Gamma_{int} = \{x_1, x_2, \cdots, x_{N-1}\}$ denote the set of interior nodes. Then

$$
\{x_0, x_1, \cdots, x_N\} = \Gamma_{int} \cup \Gamma_D \cup \Gamma_N.
$$